9.3 Theorems of Pappus and Guldinus
9.3 Theorems of Pappus and Guldinus Example 1, page 1 of 2

1. Determine the amount of paint required to paint the inside and outside surfaces of the cone, if one gallon of paint covers 300 ft$^2$.

1. The $y$ axis is the axis of revolution.

2. The generating curve is a straight line through the origin.

3. The distance to the centroid of the line is $r_C = 1.5$ ft
The length of the curve can be found from the Pythagorean theorem:

\[ L = \sqrt{(10 \text{ ft})^2 + (3 \text{ ft})^2} \]

\[ = 10.4403 \text{ ft} \]

Applying the first theorem of Pappus-Guldinus gives the area:

\[ A = 2\pi r_c L \]

\[ = 2\pi(1.5 \text{ ft})(10.4403 \text{ ft}) \]

\[ = 98.3975 \text{ ft}^2 \]

Calculate the volume of paint required:

\[ \text{Volume of paint} = 2(98.3975 \text{ ft}^2)(\frac{1 \text{ gal}}{300 \text{ ft}^2}) \]

\[ = 0.656 \text{ gal} \]

Because both the inside and outside surfaces must be painted, the value of the computed area must be doubled.
2. Determine the volume of the cone.

1. The y axis is the axis of rotation.

2. The generating area is a triangle

3. The centroid of a triangle is located one-third of the distance from the base to the opposite vertex.

4. The area of the triangle is

\[ A = \frac{1}{2} \times (10 \text{ ft}) \times (3 \text{ ft}) \]

\[ = 15 \text{ ft}^2 \]

5. Applying the second theorem of Pappus-Guldinus gives the volume:

\[ V = 2\pi r_c A \]

\[ = 2\pi (1 \text{ ft}) (15 \text{ ft}^2) \]

\[ = 94.2 \text{ ft}^3 \quad \leftarrow \text{Ans.} \]
3. Determine the area of the half-torus (half of a doughnut).

1. The axis of revolution is the x axis.

2. The generating curve is a circle.

3. The distance to the centroid is $r_c = 4\text{ m}$.

4. The length of the curve is the circumference of the circle:

   \[ L = 2\pi (1\text{ m}) = 6.2832 \text{ m} \]

5. Applying the first theorem of Pappus-Guldinus gives the area:

   \[ A = \pi r_c L = \pi (4\text{ m})(6.2832 \text{ m}) = 79.0 \text{ m}^2 \]

   ←Ans.

The angle of revolution is $\pi$, not $2\pi$, because the figure is a half-torus.
4. Determine the volume of the half-torus (half of a doughnut).

1. The axis of revolution is the x axis

2. The generating area is the region bounded by a circle

3. The distance to the centroid is \( r_c = 4 \text{ m} \)

4. The area bounded by the circle is
   \[
   A = \pi (1 \text{ m})^2 = \pi \text{ m}^2
   \]

5. Applying the second theorem of Pappus-Guldinus gives the volume:
   \[
   V = \pi r_c A = \pi (4 \text{ m})(\pi \text{ m}^2) = 39.5 \text{ m}^3 \quad \leftarrow \text{Ans.}
   \]

The angle of revolution is \( \pi \), not \( 2\pi \), because the figure is a half-torus.
5. Determine the area of the frustum of the cone.

1. The y axis is the axis of rotation

2. The generating curve is the straight line BD. The horizontal coordinate \( d \) of the lower end of the line can be found by similar triangles:

\[
\frac{d}{2m} = \frac{3m}{2m + 4m}
\]

Solving gives
\[
d = 1 \text{ m}
\]
9.3 Theorems of Pappus and Guldinus Example 5, page 2 of 2

3 The distance to the centroid is
\[ r_c = \frac{1 \text{ m} + 3 \text{ m}}{2} = 2 \text{ m} \]

4 The length of the generating curve BD is
\[ L = \sqrt{(2 \text{ m})^2 + (4 \text{ m})^2} = 4.4721 \text{ m} \]

5 Applying the first theorem of Pappus-Guldinus gives the area:
\[ A = 2\pi r_c L \]
\[ = 2\pi(2 \text{ m})(4.4721 \text{ m}) \]
\[ = 56.2 \text{ m}^2 \quad \text{Ans.} \]
6. Determine the volume of the frustum of the cone.

1. The y axis is the axis of rotation

2. The generating area is the trapezoid BDEF. The horizontal coordinate \(d\) of the lower end of the line BD can be found by similar triangles:

\[
\frac{d}{2 \text{ m}} = \frac{3 \text{ m}}{2 \text{ m} + 4 \text{ m}}
\]

Solving gives

\(d = 1 \text{ m}\)

3. The distance \(r_c\) to the centroid of the area can be calculated by dividing the crosshatched trapezoid into composite parts and using the formula

\[
r_c = \frac{\sum x_{el} A_{el}}{\sum A_{el}} \quad (1)
\]

where \(x_{el}\) is the centroidal coordinate of the part with area \(A_{el}\).
However, we can save some work by noting that the second theorem of Pappus-Guldinus involves the product of $r_c$ and the generating area $A$:

\[ V = 2\pi r_c A \]  

(2)

where $V$ is the volume of the solid of revolution.

Solving Eq. 1 for the product of $r_c$ and $\sum A_{el}$ gives

\[ r_c \sum A_{el} = \sum x_{el} A_{el} \]

$A = \text{total area} = \text{sum of individual elements of area}$

and substituting this result in Eq. 2 gives

\[ V = 2\pi \sum x_{el} A_{el} \]  

(3)

Thus we do not need to calculate $r_c$ and $A$ independently—we need only evaluate the first moment of the area, $\sum x_{el} A_{el}$. 

\[ A = \text{total area} = \sum \text{individual elements of area} \]
9.3 Theorems of Pappus and Guldinus Example 6, page 3 of 3

To calculate $\sum x_{cl}A_{cl}$, divide the trapezoidal region into the sum of a rectangle and triangle, and set up a table.

<table>
<thead>
<tr>
<th>Region</th>
<th>$A_{cl}$ (m$^2$)</th>
<th>$x_{cl}$ (m)</th>
<th>$x_{cl}A_{cl}$ (m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle</td>
<td>4</td>
<td>0.5</td>
<td>2</td>
</tr>
<tr>
<td>Triangle</td>
<td>4</td>
<td>1.6667</td>
<td>6.6668</td>
</tr>
</tbody>
</table>

$\sum x_{cl}A_{cl} = 8.6668$

Substituting the value of $\sum x_{cl}A_{cl}$ into Eq. 3 gives the volume of the solid:

$$V = 2\pi \sum x_{cl}A_{cl}$$

$$= 54.5 \text{ m}^3$$

$\leftarrow$ Ans.
7. Determine the centroidal coordinate \( r_c \) of a semicircular arc of radius \( R \), given that the area of a sphere of radius \( R \) is known to be \( 4\pi R^2 \).

1. If the semicircle is revolved around the y axis, a sphere of radius \( R \) is generated. The first theorem of Pappus-Guldinus says that the area of the sphere is given by

\[
A = 2\pi r_c L
\]

Because we already know \( A (= 4\pi R^2) \), we can solve this equation for \( r_c \) in terms of \( R \) and \( L \):

\[
r_c = \frac{A}{2\pi L}
\]

(Eq. 1)

2. The length \( L \) in Eq. 1 is the circumference of the semicircle:

\[
L = \pi R
\]

Substituting this result in Eq. 1 gives

\[
r_c = \frac{4\pi R^2}{2\pi L} = \frac{2R}{\pi}
\]

\( \leftarrow \text{Ans.} \)
9.3 Theorems of Pappus and Guldinus Example 8, page 1 of 1

8. Determine the centroidal coordinate \( r_c \) of a semicircular area of radius \( R \), given that the volume of a sphere is known to be \((4/3)\pi R^3\).

1. If the semicircular area is revolved around the \( y \) axis, a sphere of radius \( R \) is generated. The second theorem of Pappus-Guldinus says that the volume of the sphere is given by

\[
V = 2\pi r_c A
\]

Because we already know \( V = (4/3)\pi R^3 \), we can solve Eq. 1 for \( r_c \) in terms of \( A \) and \( R \):

\[
r_c = \frac{V}{2\pi A} = \frac{(4/3)\pi R^3}{2\pi A} \tag{2}
\]

2. The area \( A \) in Eq. 2 is the area bounded by the semicircle:

\[
A = \frac{\pi R^2}{2}
\]

Substituting this result in Eq. 2 gives

\[
r_c = \frac{(4/3)\pi R^3}{2\pi A} = \frac{\pi R^2}{2} \tag{Eq. 2 repeated}
\]

\[
= \frac{4R}{3\pi} \quad \text{←Ans.}
\]
9.3 Theorems of Pappus and Guldinus Example 9, page 1 of 4

9. A concrete dam is to be constructed in the shape shown. Determine the volume of concrete that would be required.

1. The y axis is the axis of revolution

2. The crosshatched region is the generating area.
The second theorem of Pappus-Guildinus gives the volume as
\[ V = 2\pi \left( \frac{40^\circ}{360^\circ} \right) r_c A \]  
where \( r_c \) is the distance to the centroid of the generating area, \( A \) is the magnitude of the area, and the factor \( 40^\circ/360^\circ \) accounts for the fact that the dam corresponds to \( 40^\circ \) rather than to a complete circle. Thus we must calculate the product \( r_c A \). This product may be found by dividing the crosshatched area into composite parts and then using the formula
\[ r_c = \frac{\sum x_{el} A_{el}}{\sum A_{el}} \] 
where \( x_{el} \) is the centroidal coordinate of the part with area \( A_{el} \). Solving for the product \( r_c A \) gives
\[ r_c A = \sum x_{el} A_{el} \]
Thus Eq. 1 can be written as
\[ V = 2\pi \left( \frac{40^\circ}{360^\circ} \right) r_c A \]  
(Eq. 1 repeated)
\[ = (2/9)\pi \sum x_{el} A_{el} \]  
(3)
and, to calculate \( V \), we have to evaluate the sum \( \sum x_{el} A_{el} \).
4 Calculate the areas and centroidal distances, and set up a table.

\[
\begin{align*}
\text{Triangle 1} & : \quad A_{el} = (1/2)(3.0 \text{ m}) \times (3.5 \text{ m}) = 5.25 \text{ m}^2 \\
\text{Rectangle} & : \quad A_{el} = (3.5 \text{ m}) \times (1 \text{ m}) = 3.5 \text{ m}^2 \\
\text{Triangle 2} & : \quad A_{el} = (1/2)(3.5 \text{ m}) \times (2 \text{ m}) = 3.5 \text{ m}^2
\end{align*}
\]

\[
x_{el} = 23 \text{ m} + (3 \text{ m})/3 = 24 \text{ m}
\]

\[
x_{el} = 20 \text{ m} + 2 \text{ m} + 0.5 \text{ m} = 22.5 \text{ m}
\]

\[
x_{el} = 20 \text{ m} + (2 \text{ m})(2/3) = 21.333 \text{ m}
\]
9.3 Theorems of Pappus and Guldinus Example 9, page 4 of 4

Table

<table>
<thead>
<tr>
<th>Region</th>
<th>( A_{el} ) (m²)</th>
<th>( x_{el} ) (m)</th>
<th>( x_{el}A_{el} ) (m³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle 1</td>
<td>5.25</td>
<td>24</td>
<td>126</td>
</tr>
<tr>
<td>Rectangle</td>
<td>3.5</td>
<td>22.500</td>
<td>78.750</td>
</tr>
<tr>
<td>Triangle 2</td>
<td>3.5</td>
<td>21.333</td>
<td>74.665</td>
</tr>
</tbody>
</table>

\[ \sum x_{el}A_{el} = 279.415 \]

Substituting the value of \( \sum x_{el}A_{el} \) into Eq. 3 gives the volume of the solid:

\[ V = \frac{2\pi}{9} \sum x_{el}A_{el} \]  
\( \text{Eq. 3 repeated} \)

\[ 279.415 \text{ m}^3 \]

\[ = 195.1 \text{ m}^3 \text{ ←Ans} \]
10. The concrete steps shown are in the shape of a quarter circle. Determine the amount of paint required to paint the steps, if one liter of paint covers 1.5 m$^2$.

1. The $y$ axis is the axis of revolution.

2. The generating curve is a series of four straight line-segments.
9.3 Theorems of Pappus and Guldinus Example 10, page 2 of 4

The first theorem of Pappus-Guildinus gives the area as

\[ A = 2 \pi \left( \frac{90^\circ}{360^\circ} \right) r_c L \]

where \( r_c \) is the distance to the centroid of the generating curve, \( L \) is the length of the area, and the factor \( \left( \frac{90^\circ}{360^\circ} \right) \) accounts for the fact that the steps are in the shape of a quarter circle. Thus we must calculate the product \( r_c L \). This product may be found from the formula for the centroid of a composite curve made of a collection of line segments:

\[ r_c = \frac{\sum x_{el} L_{el}}{\sum L_{el}} \]

where \( x_{el} \) is the centroidal coordinate of the line segment with length \( L_{el} \). Solving for the product \( r_c L \) gives

\[ r_c L = \sum x_{el} L_{el} \]

Thus Eq. 1 can be written as

\[ A = \frac{\pi}{2} r_c L \]

(Eq. 1 repeated)

\[ = \frac{\pi}{2} \sum x_{el} L_{el} \]  \hspace{1cm} (3)

and, to calculate \( A \), we have to evaluate the sum \( \sum x_{el} L_{el} \).
4. Calculate the centroidal coordinates and lengths of the line segments, and set up a table.

\[ y \]
\[ x \]
190 mm
190 mm
260 mm 260 mm

\[ y \]
\[ x \]
260 mm
380 mm
\[ x_{el} = \frac{260 \text{ mm}}{2} = 130 \text{ mm} \]

\[ y \]
\[ x \]
190 mm
190 mm
260 mm
\[ x_{el} = 260 \text{ mm} + \frac{260 \text{ mm}}{2} = 390 \text{ mm} \]

\[ y \]
\[ x \]
260 mm 260 mm
\[ L_{el} = 260 \text{ mm} \]

\[ y \]
\[ x \]
190 mm
190 mm
260 mm
\[ L_{el} = 190 \text{ mm} \]

\[ y \]
\[ x \]
190 mm
190 mm
260 mm
\[ x_{el} = 260 \text{ mm} + 260 \text{ mm} = 520 \text{ mm} \]
Substituting the value of $x_{el}A_{el}$ into Eq. 3 gives the area of the steps

\[
A = x_{el}L_{el} \quad \text{(Eq. 3 repeated)}
\]

\[
= 445 164 \text{ mm}^2
\]

Amount of paint required = $445 164 \text{ mm}^2 \times \left[1 \text{ m}/(1000 \text{ mm})\right]^2 \times [1 \text{ liter of paint}/1.5 \text{ m}^2 \text{ of covered area}]

\[
= 0.297 \text{ liter} \quad \leftarrow \text{Ans.}
\]
11. The concrete steps shown are in the shape of a quarter circle. Determine the total number of cubic meters of concrete required to construct the steps.

1. The y axis is the axis of revolution

2. The generating area is the crosshatched region shown.
The second theorem of Pappus-Guildinus gives the volume as

\[ V = 2 \left( \frac{90^\circ}{360^\circ} \right) \pi r_c A \]

\[ = \left( \frac{\pi}{2} \right) r_c A \]

where \( r_c \) is the distance to the centroid of the generating area, \( A \) is the magnitude of the area, and the factor \( (90^\circ/360^\circ) \) accounts for the fact that the steps form a quarter circle. Thus we must calculate the product \( r_c A \). This product may be found by dividing the crosshatched area into composite parts and then using the formula

\[ r_c = \frac{\sum x_{el} A_{el}}{\sum A_{el}} \]

where \( x_{el} \) is the centroidal coordinate of the part with area \( A_{el} \). Solving for the product \( r_c A \) gives

\[ r_c A = \sum x_{el} A_{el} \]

Thus Eq. 1 can be written as

\[ V = \left( \frac{\pi}{2} \right) (r_c A) \quad \text{(Eq. 1 repeated)} \]

\[ = \left( \frac{\pi}{2} \right) \sum x_{el} A_{el} \quad \text{(3)} \]

and, to calculate \( V \), we have to evaluate the sum \( \sum x_{el} A_{el} \).
4) Calculate the centroidal distances and areas and set up a table.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Region} & A_{cl} (\text{mm}^2) & x_{cl} (\text{mm}) & x_{cl}A_{cl} (\text{mm}^3) \\
\hline
\text{Rectangle 1} & 98\,800 & 130 & 12\,844\,000 \\
\text{Rectangle 2} & 49\,400 & 390 & 19\,266\,000 \\
\hline
\end{array}
\]

\[
\sum x_{cl}A_{cl} = 32\,110\,000
\]

5) Substituting the value of \(\sum x_{cl}A_{cl}\) into Eq. 3 gives the volume of the solid

\[
V = \frac{\pi}{2} \sum x_{cl}A_{cl} = 32\,110\,000 \text{ mm}^3
\]

\[
= 50\,438\,270 \text{ mm}^3
\]

\[
= 0.0504 \text{ m}^3 \quad \text{Ans.}
\]
12. Determine the mass of the steel V-belt pulley shown. The density of the steel is 7840 kg/m³.

1. The x axis is the axis of revolution:

2. The crosshatched area is the generating area for the right half of the pulley.
The second theorem of Pappus-Guildinus gives the volume as

\[ V = 2\pi r_c A \]  \hspace{1cm} (1)

where \( r_c \) is the distance to the centroid of the generating area, and \( A \) is the magnitude of the area. Thus we must calculate the product \( r_c A \). This product may be found by dividing the crosshatched area into composite parts and then using the formula

\[ r_c = \frac{\sum y_{el} A_{el}}{\sum A_{el}} \]  \hspace{1cm} (2)

where \( x_{el} \) is the centroidal coordinate of the part with area \( A_{el} \). Solving for the product \( r_c A \) gives

\[ r_c A = \sum y_{el} A_{el} \]

Thus Eq. 1 can be written as

\[ V = 2\pi (r_c A) \] \hspace{1cm} (Eq. 1 repeated)

\[ = 2\pi \sum y_{el} A_{el} \]  \hspace{1cm} (3)

and, to calculate \( V \), we have to evaluate the sum \( \sum y_{el} A_{el} \).
9.3 Theorems of Pappus and Guldinus Example 12, page 3 of 4

4 Calculate the centroidal distances and areas for the rectangles and triangle, and set up a table.

\[
A_{el} = (1/2)(12 \text{ mm})(15 \text{ mm})
\]
\[
= 90 \text{ mm}^2
\]

\[
A_{el} = (4 \text{ mm})(15 \text{ mm})
\]
\[
= 60 \text{ mm}^2
\]

Generating area divided into two rectangles, and a triangle

\[
yel = 10 \text{ mm} + 25/2 \text{ mm}
\]
\[
= 22.5 \text{ mm}
\]

\[
yel = 35 \text{ mm} + 15/3 \text{ mm}
\]
\[
= 40 \text{ mm}
\]

\[
yel = 35 \text{ mm} + 15/2 \text{ mm}
\]
\[
= 42.5 \text{ mm}
\]

\[
A_{el} = (21 \text{ mm})(25 \text{ mm})
\]
\[
= 525 \text{ mm}^2
\]

\[
A_{el} = (1/2)(12 \text{ mm})(15 \text{ mm})
\]
\[
= 90 \text{ mm}^2
\]
9.3 Theorems of Pappus and Guldinus Example 12, page 4 of 4

5) Table

<table>
<thead>
<tr>
<th>Region</th>
<th>$A_{el}$ (mm$^2$)</th>
<th>$y_{el}$ (mm)</th>
<th>$y_{el}A_{el}$ (mm$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle 1</td>
<td>525</td>
<td>22.5</td>
<td>11 812.5</td>
</tr>
<tr>
<td>Triangle</td>
<td>90</td>
<td>40.0</td>
<td>3 600.0</td>
</tr>
<tr>
<td>Rectangle 2</td>
<td>60</td>
<td>42.5</td>
<td>2 550.0</td>
</tr>
</tbody>
</table>

$\sum y_{el}A_{el} = 17 962.5$

6) Substituting the value of $\sum y_{el}A_{el}$ into Eq. 3 gives half the volume of the V-belt pulley

$$V = 2\pi \sum y_{el}A_{el}$$

$$= 2\pi \times 17 962.5 \text{ mm}^3$$

$$= 112 861.7 \text{ mm}^3 = 0.000 112 861 7 \text{ m}^3$$

7) Total mass of the V-belt = total volume $\times$ density

$$= (2 \times 0.000 112 861 7 \text{ m}^3) \times 7840 \text{ kg/m}^3$$

$$= 1.770 \text{ kg}$$

Double the half-volume.
13. Determine the area of the surface of revolution generated by rotating the curve \( y = z^4, \ 0 \leq z \leq 1 \) m, about the z axis.

1. The z axis is the axis of revolution.
2. The generating curve is
   \[
y = z^4, \ 0 \leq z \leq 1
   \]
The first theorem of Pappus-Guldinus gives the area of the surface of revolution as

\[ A = 2\pi r_c L \]  

(1)

where \( r_c \) is the distance to the centroid of the generating curve, and \( L \) is the length of the curve. Thus we must calculate the product \( r_c L \). This product may be found by considering the equation for the centroidal coordinate:

\[ r_c = \frac{\int y_{el} \, dL}{\int dL} \]

where \( dL \) is an increment of curve length, and \( y_{el} \) is the coordinate of the increment. Solving for the product \( r_c L \) gives

\[ r_c L = \int y_{el} \, dL \]

Thus Eq. 1 can be written as

\[ A = 2\pi r_c L \]

\[ = 2\pi \int y_{el} \, dL \]  

(2)

and, to calculate \( A \), we have to evaluate a single integral, \( \int y_{el} \, dL \).
9.3 Theorems of Pappus and Guldinus Example 13, page 3 of 3

To evaluate the integral \( \int y_{el} \, dL \) in Eq. 2, we use the equation of the curve,

\[
y = z^4
\]

(3)

to express \( dL \) as a function of \( z \). Thus

\[
dL = \sqrt{(dy)^2 + (dz)^2} = \sqrt{\left(\frac{dy}{dz}\right)^2 + 1} \, dz
\]

(4)

and differentiating Eq. 3 gives

\[
\frac{dy}{dz} = 4z^3
\]

so Eq. 4 can be written as

\[
dL = \sqrt{(4z^3)^2 + 1} \, dz
\]

(5)

Substitute this expression for \( dL \) into Eq. 2

\[
A = 2\pi \int y_{el} \, dL \quad \text{(Eq. 2 repeated)}
\]

(5)

Evaluating the integral by use of the integral function key on a calculator gives

\[
A = 3.44 \, m^2 \quad \text{Ans.}
\]
14. Determine the volume of the solid of revolution generated by rotating the curve $y = z^4$, $0 \leq z \leq 1$ m, about the z axis.

1. The $z$ axis is the axis of rotation.
2. The generating area is the area under the $y = z^4$ curve.
The second theorem of Pappus-Guildinus gives the volume as

\[ V = 2\pi r_c A \]  

(1)

where \( r_c \) is the distance to the centroid of the generating area, and \( A \) is the magnitude of the area. Thus we must calculate the product \( r_c A \). This product may be found by considering the equation for the centroidal distance:

\[ r_c = \frac{\int y_{el} \, dA}{\int dA} \]

where \( dA \) is an increment of area, and \( y_{el} \) is the coordinate of the centroid of the incremental region. Solving for the product \( r_c A \) gives

\[ r_c A = \int y_{el} \, dA \]

Thus Eq. 1 can be written as

\[ V = 2\pi (r_c A) \]

\[ = 2\pi \int y_{el} \, dA \]  

(2)

and, to calculate \( V \), we have to evaluate a single integral, \( \int y_{el} \, dA \).
To evaluate the integral \( \int y \, dL \) in Eq. 2, we use the equation of the curve,

\[
y = z^4
\]

to express \( dA \) as a function of \( z \). Thus

\[
dA = y \, dz = z^4 \, dz
\]

Substitute this expression for \( dA \) into Eq. 2

\[
A = 2\pi \int y \, dz
\]

(Eq. 2 repeated)

\[
= 2\pi \int_0^1 (z^4/2)(z^4) \, dz
\]

Evaluating the integral by use of the integral function key on a calculator gives

\[
V = 0.349 \, m^2
\]

\( \text{Ans.} \)
15. A pharmaceutical company plans to put a coating 0.01 mm thick on the outside of the pill shown. Determine the amount of coating material required.

1. The x axis is the axis of revolution.

2. The generating curve for half of the pill surface is a composite curve consisting of one straight line and a circular arc. By symmetry, the total surface area of the pill will be two times the area generated by the curve above.
The first theorem of Pappus-Guldinus gives the area as
\[ A = 2\pi r_c L \quad (1) \]
where \( r_c \) is the distance to the centroid of the generating curve and \( L \) is the length of the curve. Thus we must calculate the product \( r_c L \). This product may be found by dividing the curve into composite parts and then using the formula
\[ r_c = \frac{\sum y_{el} L_{el}}{\sum L_{el}} \quad (2) \]
where \( x_{el} \) is the centroidal coordinate of the part with length \( L_{el} \). Solving for the product \( r_c L \) gives
\[ r_c L = \sum y_{el} L_{el} \]
Thus Eq. 1 can be written as
\[ A = 2\pi (r_c L) \quad \text{(Eq. 1 repeated)} \]
\[ = 2\pi \sum y_{el} L_{el} \quad (3) \]
and, to calculate \( A \), we have to evaluate the sum \( \sum y_{el} L_{el} \).
9.3 Theorems of Pappus and Guldinus Example 15, page 3 of 5

4) Calculate the centroidal distances and the lengths, and set up a table.

For the straight line, the length and coordinate of the centroid are easily calculated.

\[ L_{el} = 0.75 \text{ mm} \]

\[ y_{el} = 3.5 \text{ mm} \]
9.3 Theorems of Pappus and Guldinus Example 15, page 4 of 5

6 To calculate $\Sigma x_{el}L_{el}$ for the arc, use the information shown below, which has been taken from a table of properties of common geometric shapes.

For our particular arc, $r = 20$ mm and

$$\alpha = (1/2) \sin^{-1}(3.5/20)$$

$$= 0.08795 \text{ rad}$$

Thus

Length $= 2\alpha r$

$$= 2(0.08795) \times 20 \text{ mm}$$

$$= 3.5180 \text{ mm}$$

$$r_{arc} = (r \sin \alpha)/\alpha$$

$$= (20 \text{ mm})(\sin 0.08795)/(0.08795)$$

$$= 19.9742 \text{ mm}$$

$y_{el} = r_{arc} \sin \alpha$

$$= 19.9742 \text{ mm} \sin 0.08795$$

$$= 1.7545 \text{ mm}$$
9.3 Theorems of Pappus and Guldinus Example 15, page 5 of 5

Substituting the value of $y_{el}A_{el}$ into Eq. 3 gives the area of the solid

$$A = 2 \times 2\pi \sum y_{el}L_{el}$$

$$= 110.5501 \text{ mm}^2$$

where a factor of 2 has been inserted to account for the fact that we took advantage of symmetry to calculate the area of only half of the body.

Amount of coating material required = $110.5501 \text{ mm}^2 \times 0.01 \text{ mm}$

$$= 1.106 \text{ mm}^3$$

← Ans.
9.3 Theorems of Pappus and Guldinus Example 16, page 1 of 4

16. Determine the volume of the funnel.

1. The y axis is the axis of revolution

2. The generating area is the crosshatched area shown.
The second theorem of Pappus-Guildinus gives the volume as

\[ V = 2\pi r_c A \]  

(1)

where \( r_c \) is the distance to the centroid of the generating area and \( A \) is the magnitude of the area. Thus we must calculate the product \( r_c A \). This product may be found by dividing the cross-hatched area into composite parts and then using the formula

\[ r_c = \frac{\sum x_{el} A_{el}}{\sum A_{el}} \]

(2)

where \( x_{el} \) is the centroidal coordinate of the part with area \( A_{el} \). Solving for the product \( r_c A \) gives\n
\[ r_c A = \sum x_{el} A_{el} \]

Thus Eq. 1 can be written as

\[ V = 2\pi (r_c A) \]

(Eq. 1 repeated)

\[ = 2\pi \sum x_{el} A_{el} \]

(3)

and, to calculate \( V \), we have to evaluate the sum \( \sum x_{el} A_{el} \).
To calculate $\Sigma x_{el}A_{el}$, divide the area into the sum of two rectangles and two triangles, and set up a table.

\[
\begin{align*}
\text{Rectangle 1} & : x_{el} = 2.5 \text{ mm} \\
A_{el} & = (5 \text{ mm})(60 \text{ mm}) \\
& = 300 \text{ mm}^2 \\
\text{Triangle 1} & : x_{el} = 5 \text{ mm} + (5 \text{ mm})/3 \\
& = 6.6667 \text{ mm} \\
A_{el} & = (1/2)(5 \text{ mm})(60 \text{ mm}) \\
& = 150 \text{ mm}^2 \\
\text{Rectangle 2} & : x_{el} = 2.5 \text{ mm} \\
A_{el} & = (2.5 \text{ mm})(70 \text{ mm}) \\
& = 175 \text{ mm}^2 \\
\text{Triangle 2} & : x_{el} = 2.5 \text{ mm} + (2.5 \text{ mm})/3 \\
& = 3.3333 \text{ mm} \\
A_{el} & = (1/2)(2.5 \text{ mm})(70 \text{ mm}) \\
& = 87.5 \text{ mm}^2
\end{align*}
\]
9.3 Theorems of Pappus and Guldinus Example 16, page 4 of 4

Substituting the value of \( x_{el} A_{el} \) into Eq. 3 gives the volume of the funnel

\[
V = x_{el} A_{el} \text{ (Eq. 3 repeated)}
\]

\[
= 14,200 \text{ mm}^3 \quad \text{Ans.}
\]
17. A satellite dish is shaped in the form of a paraboloid of revolution to take advantage of the geometrical fact that all signals traveling parallel to the axis of the paraboloid are reflected through the focus. Determine the amount, in m$^2$, of reflecting material required to cover the inside surface of the dish.

The x axis is the axis of revolution

The generating curve is a parabola.

The general form for a parabola with vertex on the x axis is

$$x = ay^2 + b$$

Evaluating this equation at the points (−0.2, 0) and (0, 0.3) gives the equations

$$-0.2 = a(0)^2 + b \quad (1)$$
and

$$0 = a(0.3)^2 + b$$

Solving for a and b and substituting back in Eq. 1 gives

$$x = 0.2222y^2 - 0.2 \quad (2)$$
9.3 Theorems of Pappus and Guldinus Example 17, page 2 of 3

4. Apply the first theorem of Pappus-Guldinus to calculate the surface area of the dish:

\[ A = 2\pi r_c L \]  \hspace{1cm} (3)

Thus we must calculate the *product* of the length of the parabolic curve and its centroidal coordinate. The product may be found by considering the equation for the centroidal coordinate:

\[ r_c = \frac{\int y_{el} \, dL}{\int dL} \]

Solving for the product \( r_c L \) gives

\[ r_c L = \int y_{el} \, dL \]

Thus Eq. 3 can be written as

\[ A = 2\pi (r_c L) \]

4 = \[ 2\pi \int y_{el} \, dL \]  \hspace{1cm} (4)

5. To evaluate the integral \( \int y_{el} \, dL \) in Eq. 4, we have to use the equation of the parabola,

\[ x = 0.2222y^2 - 0.2 \]  \hspace{1cm} (Eq. 2 repeated)

to express \( dL \) as a function of \( y \). Thus

\[ dL = \sqrt{(dx)^2 + (dy)^2} \]

Differentiating Eq. 2 gives

\[ \frac{dx}{dy} = 0.4444y, \]

so Eq. 5 can be written as

\[ dL = \sqrt{(0.4444y)^2 + 1} \, dy \]  \hspace{1cm} (5)
9.3 Theorems of Pappus and Guldinus Example 17, page 3 of 3

Noting that $y = y_{el}$ and also using Eq. 6 to replace $dL$ in Eq. 4 gives

$$A = 2\pi \int_{y_{el}}^{y} dL \quad \text{(Eq. 4 repeated)}$$

$$= 2\pi \int_{0}^{0.3} y \sqrt{(0.4444y)^2 + 1} \, dy$$

Using the integral function on a calculator gives

$$A = 0.284 \text{ m}^2 \quad \text{Ans}$$
18. Determine the amount of coffee that the coffee mug holds when full to the brim. The radius of the rounded corners and the rim is 15 mm.

1. The y axis is the axis of revolution, and the generating area is the cross-hatched area shown.

2. The distance $r_c$ to the centroid of the area can be calculated by dividing the crosshatched area into composite parts and using the formula

$$r_c = \frac{\sum x_{el} A_{el}}{\sum A_{el}}$$

where $x_{el}$ is the centroidal coordinate of the part with area $A_{el}$.
However, we can save some work by noting that the second theorem of Pappus-Guldinus involves the product of \( r_c \) and the generating area \( A \):

\[
V = 2\pi r_c A \tag{2}
\]

where \( V \) is the volume of solid of revolution.

Solving Eq. 1 for the product gives

\[
r_c \sum_{i} A_{el} = \sum_{i} x_{el} A_{el} \tag{3}
\]

and substituting this result in Eq. 2 gives

\[
V = 2\pi \sum_{i} x_{el} A_{el} \tag{3}
\]

Thus we do not need to calculate \( r_c \) and \( A \) independently—we need only evaluate the first moment of the area, \( \sum x_{el} A_{el} \).
To calculate $\sum x_\theta A_\theta$, divide the area into the algebraic sum of a rectangle, two squares, and two quarter circles, and set up a table.

For clarity, the areas near the rim and rounded corner have been drawn disproportionately large.
9.3 Theorems of Pappus and Guldinus Example 18, page 4 of 6

7) Calculate the areas and centroidal coordinates of the rectangle and the squares.

Original area

Radius = 15 mm

\[ A_{el} = (15 \text{ mm})^2 = 225 \text{ mm}^2 \]

\[ A_{el} = (40 \text{ mm})(90 \text{ mm}) = 3600 \text{ mm}^2 \]

\[ x_{el} = 40 \text{ mm} + (15 \text{ mm})/2 = 47.5 \text{ mm} \]

\[ x_{el} = 40 \text{ mm} - (15 \text{ mm})/2 = 32.5 \text{ mm} \]

\[ A_{el} = (15 \text{ mm})^2 = 225 \text{ mm}^2 \]
Calculate the area and centroidal coordinates of the quarter-circular regions.

\begin{align*}
x_{cl} &= 40 \text{ mm} + 15 \text{ mm} - 6.3662 \text{ mm} \\
&= 48.6338 \text{ mm}
\end{align*}

\begin{align*}
x_{cl} &= 40 \text{ mm} - 15 \text{ mm} \\
&= 31.3662 \text{ mm}
\end{align*}

A table of properties of planar regions gives the information shown below.

\begin{align*}
A &= \frac{\pi r^2}{4} \\
C &= \frac{4r}{3\pi}
\end{align*}

In our particular problem, \( r = 15 \text{ mm} \), so the distance to the centroid is

\[ \frac{4r}{3\pi} = \frac{4(15 \text{ mm})}{3\pi} = 6.3662 \text{ mm}^4 \]

Also, the area is

\[ A_{cl} = \frac{\pi (15 \text{ mm})^2}{4} = 176.7146 \text{ mm}^2 \]
Substituting the value of \( x_{el}A_{el} \) into Eq. 3 gives the volume of the solid

\[
V = 2\pi \sum x_{el}A_{el} \approx 72323.563 \text{ mm}^3
\]

\[= 454000 \text{ mm}^3 \ \ \ \ \ \longleftrightarrow \ \ \ \text{Ans.}
\]
19. Determine the capacity of the small bottle of lotion if the bottle is filled half way up the neck.

1. The y axis is the axis of revolution.

2. The generating area is the crosshatched area shown.

Half of the neck is filled:
\[
\frac{(5 \text{ mm})}{2}
\]
The second theorem of Pappus-Guildinus gives the volume as

\[ V = 2\pi r_c A \]  \hspace{1cm} (1)

where \( r_c \) is the distance to the centroid of the generating area, and \( A \) is the magnitude of the area. Thus we must calculate the product \( r_c A \). This product may be found by dividing the cross-hatched area into composite parts and then using the formula

\[ r_c = \frac{\sum x_{el} A_{el}}{\sum A_{el}} \]  \hspace{1cm} (2)

where \( x_{el} \) is the centroidal coordinate of the part with area \( A_{el} \). Solving for the product \( r_c A \) gives

\[ r_c A = \sum x_{el} A_{el} \]

Thus Eq. 1 can be written as

\[ V = 2\pi (r_c A) \]  \hspace{1cm} (Eq. 1 repeated)

\[ = 2\pi \sum x_{el} A_{el} \]  \hspace{1cm} (3)

and, to calculate \( V \), we have to evaluate the sum \( \sum x_{el} A_{el} \).
Before the areas and centroidal coordinates can be found, we first must find the distances and angles shown below.

**Example 19**

\[\begin{align*}
\text{DC} &= \sqrt{(20 \text{ mm})^2 - (17.5 \text{ mm})^2} \\
&= 9.6825 \text{ mm} \\
\angle \text{DOC} &= \cos^{-1} \frac{17.5}{20} \\
&= 28.9550^\circ \\
\text{EB} &= \sqrt{(20 \text{ mm})^2 - (15 \text{ mm})^2} \\
&= 13.2288 \text{ mm} \\
\angle \text{EOB} &= \cos^{-1} \frac{15}{20} \\
&= 41.4096^\circ
\end{align*}\]
The areas and centroidal coordinates of the rectangle and triangles can now be calculated.

Triangle 1

\[ A_{el} = (2.5 \text{ mm})(9.6825 \text{ mm}) \]
\[ = 24.2062 \text{ mm}^2 \]

Triangle 2

\[ A_{el} = (1/2)(17.5 \text{ mm})(9.6825 \text{ mm}) \]
\[ = 84.7219 \text{ mm}^2 \]

\[ x_{el} = 9.6825 \text{ mm}/3 \]
\[ = 3.2275 \text{ mm} \]

\[ x_{el} = (13.2288 \text{ mm})/3 \]
\[ = 4.4096 \text{ mm} \]

Rectangle

\[ A_{el} = (1/2)(15 \text{ mm})(13.2288 \text{ mm}) \]
\[ = 99.2160 \text{ mm}^2 \]
To calculate the area and centroidal coordinate of the circular sector, we can use the information shown below, which has been taken from a table of properties of planar regions. Note that \( \theta \) in the formula equals half the angle of the arc.

To calculate the area and centroidal coordinate of the circular sector, we can use the information shown below, which has been taken from a table of properties of planar regions. Note that \( \theta \) in the formula equals half the angle of the arc.

\[
\theta = \frac{(180^\circ - \angle DOC - \angle EOB)}{2}
\]

by Eq. 4

\[
= \frac{(180^\circ - 28.9550^\circ - 41.4096^\circ)}{2}
\]

\[
= 54.8177^\circ
\]

\[
\alpha = \theta + \angle EOB - 90^\circ
\]

\[
= 54.8177^\circ + 41.4096^\circ - 90^\circ
\]

\[
= 6.2273^\circ
\]

\[
r_c = \frac{2r \sin \theta}{3\theta}
\]

\[
= \frac{2(20 \text{ mm}) \sin 54.8177^\circ}{3(54.8177^\circ \times \pi/180^\circ)}
\]

\[
= 11.3903 \text{ mm}
\]

\[
x_{el} = r_c \cos \alpha
\]

\[
= (11.3903 \text{ mm}) \cos 6.2273^\circ
\]

\[
= 11.3231 \text{ mm}
\]
9.3 Theorems of Pappus and Guldinus Example 19, page 6 of 6

Substituting the value of \( x_{el}A_{el} \) into Eq. 3 gives the capacity of the bottle:

\[
V = 2\pi \sum x_{el}A_{el}
\]

\[
= 32400 \text{ mm}^3 \quad \leftarrow \text{Ans.}
\]

<table>
<thead>
<tr>
<th>Region</th>
<th>( A_{el} ) (mm(^2))</th>
<th>( x_{el} ) (mm)</th>
<th>( x_{el}A_{el} ) (mm(^3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle 1</td>
<td>84.7219</td>
<td>3.2275</td>
<td>273.4399</td>
</tr>
<tr>
<td>Triangle 2</td>
<td>99.2160</td>
<td>4.4096</td>
<td>437.5029</td>
</tr>
<tr>
<td>Rectangle</td>
<td>24.2062</td>
<td>4.8412</td>
<td>117.1871</td>
</tr>
<tr>
<td>Circular sector</td>
<td>382.6997</td>
<td>11.3231</td>
<td>4333.3470</td>
</tr>
</tbody>
</table>

\( \sum x_{el}A_{el} = 5161.4768 \)