10.1 Moments of Inertia by Integration
1. Determine the moment of inertia of the rectangle about the x axis, which passes through the centroid C.
We want to evaluate

\[ I_x = \int y^2 \, dA \]

where the differential element \( dA \) is located a distance \( y \) from the \( x \) axis (\( y \) must have the same value throughout \( dA \)).

\[ dA = \text{area of rectangle} = b \times dy \]

\[ I_x = \int_{-h/2}^{h/2} y^2 (b \times dy) \]

\[ = \frac{by^3}{3} \bigg|_{-h/2}^{h/2} \]

\[ = \frac{bh^3}{12} \]  \rightarrow \text{Ans.}
2. Determine the moment of inertia of the rectangle about its base, which coincides with the x axis.
We want to evaluate
\[ I_x = \int y^2 \, dA \]
where the differential element of area \( dA \) is located a distance \( y \) from the \( x \)-axis (\( y \) must have the same value throughout the element \( dA \)).

\[ dA = \text{area of rectangle} = b \times dy \]

\[ I_x = \int y^2 dA \]
\[ = \int_0^h y^2 (b \times dy) \]
\[ = \frac{by^3}{3} \bigg|_0^h \]
\[ = \frac{bh^3}{3} \]
\( \leftarrow \text{Ans.} \)
3. Determine the moment of inertia of the right triangle about the x axis.
10.1 Moments of Inertia by Integration Example 3, page 2 of 3

We want to evaluate

\[ I_x = \int y^2 \, dA \]

where \( y \) has the same value throughout differential element \( dA \).

Equation of line

\[ y = \text{slope} \times x + \text{intercept} \]

\[ = -\left( \frac{h}{b} \right) x + h \]  \hspace{1cm} (1)

Solve for \( x \) to get

\[ x = \left( -\frac{b}{h} \right) y + b \]

so,

\[ dA = x \times dy \]

\[ = \left( -\frac{b}{h} \right) y + b \, dy \]

Since we will integrate with respect to \( y \), we must replace \( x \) by a function of \( y \).
\[ I_x = \int y^2 \, dA \]

\[ = \int_0^h y^2 \left( -\frac{b}{h} y + b \right) \, dy \]
\[ = b \int_0^h \left( -\frac{y^3}{h} + y^2 \right) \, dy \]
\[ = bh^3 \left[ -\frac{y^4}{4h} + \frac{y^3}{3} \right]_0^h \]
\[ = bh^3 \left[ -\frac{1}{4} + \frac{1}{3} \right] \]
\[ = \frac{bh^3}{12} \quad \text{Ans.} \]
10.1 Moments of Inertia by Integration Example 4, page 1 of 3

4. Determine the moments of inertia of the area bounded by an ellipse about the x and y axes.

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]
10.1 Moments of Inertia by Integration Example 4, page 2 of 3

1. We want to evaluate

\[ I_x = \int y^2 \, dA \]

where \( y \) has the same value throughout the differential element \( dA \).

2. \( x = \) half the length of the differential element, so \( 2x = \) entire length

3. \( dA = \) area of rectangle

\[ = 2x \times dy \]

4. Since we will integrate with respect to \( y \), we must replace \( x \) by a function of \( y \)

5. \[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

Solve for \( x \) to get

\[ x = \pm a \sqrt{1 - \left(\frac{y}{b}\right)^2} \]

6. Since \( x \) locates a point to the right of the \( y \) axis, choose the plus sign:

\[ x = +a \sqrt{1 - \left(\frac{y}{b}\right)^2} \]  \hfill (1)
10.1 Moments of Inertia by Integration Example 4, page 3 of 3

7. Using Eq. 1 in the expression for dA gives:

\[ dA = 2x \times dy \]

\[ = 2a \sqrt{1 - (y/b)^2} \times dy \]

8. \[ I_x = \int y^2 \, dA \]

9. Limits of integration from bottom to top of ellipse

\[ = \int_{-b}^{b} y^2(2a \sqrt{1 - (y/b)^2}) \, dy \]

\[ = \frac{\pi ab^3}{4} \leftarrow \text{Ans.} \]

10. Evaluate the integral either with a calculator that does symbolic integration or use a table of integrals.

11. To calculate \( I_y \), use symmetry in the following way: in all the above equations, replace x's by y's, original y's by x's, a's by b's and original b's by a's. Then the result would be

\[ I_y = \frac{\pi a^3 b}{4} \leftarrow \text{Ans.} \]
5. Determine the moments of inertia of the crosshatched area about the x and y axes.

\[ y = 4 - x^2 \]
We want to evaluate 

\[ I_x = \int y^2 \, dA \]

where \( y \) has the same value throughout the differential element \( dA \).

\[ dA = \text{area of rectangle} = x \times dy \]

Since we will integrate with respect to \( y \), we must replace \( x \) by a function of \( y \).

\[ y = 4 - x^2 \]

Solving for \( x \) to get

\[ x = \pm \sqrt{4 - y} \]

Since \( x \) locates a point to the right of the \( y \) axis, choose the plus sign:

\[ x = +\sqrt{4 - y} \quad (1) \]
Using Eq. 1 in the expression for $dA$ gives

\[ dA = x \times dy \]
\[ = \sqrt{4 - y} \ dy \]

\[ I_x = \int y^2 \ dA \]
\[ = \int_0^4 y^2 \sqrt{4 - y} \ dy \]

\( y = 4 \) at the top of the crosshatched area

Use the integral function on your calculator

\[ = 19.50 \text{ ft}^4 \]  \[ \text{Ans.} \]
Next we want to evaluate

$$I_y = \int x^2 \, dA$$

where $x$ has the same value throughout the differential element $dA$. Thus, we choose a vertical rectangular strip.

\[ dA = \text{area of rectangle} = y \times dx = (4 - x^2) \, dx \]

\[ I_y = \int x^2 \, dA = \int_0^2 x^2(4 - x^2) \, dx = 4.27 \, ft^4 \quad \leftarrow \text{Ans.} \]
6. Determine the moment of inertia of the crosshatched area about the y axis.

Scales on the x and y axes are not the same.
We want to evaluate

\[ I_y = \int x^2 \, dA \]

where \( x \) has the same value throughout the differential element \( dA \). Thus, we choose a vertical rectangular strip.

\[ dA = \text{area of rectangle} = y \, dx \]

But this approach \emph{won't work} because we can't express \( y \) as a function of \( x \).
4. An alternative approach is to use a horizontal rectangular strip and employ the equation for the moment of inertia of a rectangle about its base (BB):

\[ I_{BB} = \frac{bh^3}{3} \quad (1) \]

5. Applying Eq. 1 to the differential element gives the differential moment of inertia.

\[ dI_y = \frac{bh^3}{3} \cdot \frac{(dy)x^3}{3} \]

6. Replacing \( x \) by the function of \( y \) gives

\[ dI_y = \frac{1}{3} x^3 dy \]

\[ = \frac{1}{3} (2y^6 - 50y^5 - y^3 + 100)^3 dy \]

7. \( I_y = \int dI_y = \int_0^{1.156} \frac{1}{3} (2y^6 - 50y^5 - y^3 + 100)^3 dy \]

\[ = 2.72 \times 10^5 \text{ m}^4 \quad \text{Ans.} \]

8. Use the integral function on a calculator to evaluate this integral.
7. Determine the moment of inertia of the crosshatched area about the x axis.
We want to evaluate 

\[ I_x = \int y^2 \, dA \]

where \( y \) has the same value throughout the differential element \( dA \). Thus it appears that we should use a horizontal rectangular strip.

But using a horizontal strip is awkward—we would have to use three different expressions for \( dA \), depending on the position of the strip.

A better approach is to use a vertical strip and then apply the parallel-axis theorem to the strips.

The moment of inertia of a strip of width \( b \equiv dx \) is then

\[ dI_x = \left( \frac{1}{12} h^3 + hd^2 \right) dx \quad (1) \]
10.1 Moments of Inertia by Integration Example 7, page 3 of 3

5. The y-coordinate of the element centroid is the average of \( y_1 \) and \( y_2 \)

\[ d = \frac{y_{el}}{(x, y_{el})} = \frac{1}{2}(y_1 + y_2) \]

6. Eq. 1 becomes

\[ dI_x = \left( \frac{h^3}{12} + hd^2 \right) dx \]

\[ = \left\{ \frac{1}{12} \left( y_2 - y_1 \right)^3 + \left( y_2 - y_1 \right) \left[ \frac{1}{2} \left( y_1 + y_2 \right) \right]^2 \right\} dx \]  

Since the point \((x, y_2)\) lies on the curve \( y = 9 - x^2 \), we can substitute

\[ y_2 = 9 - x^2 \]

in Eq. 2. Similarly, since \((x, y_1)\) satisfies \( y = 3x - x^2 \), we can substitute

\[ y_1 = 3x - x^2 \]

in Eq. 2. Thus Eq. 2 becomes

\[ dI_x = \left\{ \frac{1}{12} \left( y_2 - y_1 \right)^3 + \left( y_2 - y_1 \right) \left[ \frac{1}{2} \left( y_1 + y_2 \right) \right]^2 \right\} dx \]

\[ = \left\{ \frac{1}{12} \left[ \left( 9 - x^2 \right) - (3x - x^2) \right]^3 + \left( 9 - x^2 \right) \right\} \]

\[ = \left\{ \frac{1}{12} \left( 9 - 3x \right)^3 + \left( 9 - 3x \right) \left[ \frac{9}{2} + \frac{3}{2} x - x^2 \right] \right\} dx \]

7. \( I_x = \int dI_x \)

\[ = \int \left\{ \frac{1}{12} \left( 9 - 3x \right)^3 + \left( 9 - 3x \right) \left[ \frac{9}{2} + \frac{3}{2} x - x^2 \right] \right\} dx \]

\[ = 328 \text{ m}^4 \quad \text{Ans.} \]

8. Enter this expression directly into the integral function of a calculator.
8. Determine the moment of inertia of the crosshatched area about the y axis.

\[ xy = 1 \]
\[ y = \begin{cases} 
 2x & \text{for } 4 \text{ m} \\
 0 & \text{for } 1 \text{ m} \\
 -2x & \text{for } 1/4 \text{ m} \\
 0 & \text{for } 1/2 \text{ m} 
\end{cases} \]
10.1 Moments of Inertia by Integration Example 8, page 2 of 4

1) We want to evaluate

\[ I_y = \int x^2 \, dA \]

where \( x \) has the same value throughout the differential element \( dA \). Thus it appears that we should use *vertical* differential strips.

2) But using a vertical strip is awkward—we would have to use three different expressions for \( dA \), depending on the location of the strip.
A better approach is to use a horizontal strip and then apply the parallel-axis theorem to the strip.

Parallel-axis theorem for a general region

\[ I_y = I_c + Ad^2 \]

For a rectangle in particular,

\[ I_y = \frac{1}{12} bh^3 + (b \times h)d^2 \]

\[ = \left( \frac{1}{12} h^3 + hd^2 \right) b \]

The moment of inertia of a strip of width \( b = dy \) is then

\[ dI_y = \left( \frac{1}{12} h^3 + hd^2 \right) dy \] (1)
10.1 Moments of Inertia by Integration Example 8, page 4 of 4

From the figure, we see that Eq. 1 can be written as

\[ \text{d}I_y = \left\{ \frac{1}{12}h^3 + hd^2 \right\} \text{dy} \]  
(Eq. 1 repeated)

\[ = \left\{ \frac{1}{12} (x_2 - x_1)^3 + (x_2 - x_1)\left[ \frac{1}{2} (x_2 + x_1) \right]^2 \right\} \text{dy} \]  
(2)

Since the point \((x_1,y)\) lies on the line \(y = -2x\), we can solve for \(x\) and substitute

\[ x_1 = -\frac{y}{2} \]

in Eq. 2. Similarly, since \((x_2,y)\) lies on the curve \(xy = 1\), we can substitute

\[ x_2 = \frac{1}{y} \]

in Eq. 2. Thus Eq. 2 becomes

\[ \text{d}I_y = \left\{ \frac{1}{12} (x_2 - x_1)^3 + (x_2 - x_1)\left[ \frac{1}{2} (x_2 + x_1) \right]^2 \right\} \text{dy} \]

\[ = \left\{ \frac{1}{12} \left( \frac{1}{y} - \left( -\frac{y}{2} \right) \right)^3 + \left( \frac{1}{y} - \left( -\frac{y}{2} \right) \right)\left[ \frac{1}{2} \left( \frac{1}{y} + \left( -\frac{y}{2} \right) \right) \right]^2 \right\} \text{dy} \]

\[ = \left\{ \frac{1}{12} \left( \frac{1}{y} + \frac{y}{2} \right)^3 + \left( \frac{1}{y} + \frac{y}{2} \right)\left[ \frac{1}{2y} - \frac{y}{4} \right]^2 \right\} \text{dy} \]

Enter this expression directly into the integral function of a calculator

\[ = 2.81 \text{ m}^4 \quad \text{Ans.} \]
9. Determine the moment of inertia of the crosshatched area about the x axis.

\[
y = 10e^{-x^2}
\]

\[
y = -10e^{-x^2}
\]

1 m

1.5 m
10.1 Moments of Inertia by Integration Example 9, page 2 of 3

1. We want to evaluate
   \[ I_x = \int y^2 \, dA \]

2. Because of the shape of the region, the integral has to be evaluated over two sub-regions, D and E.

Region D

\[ y(1) = 10e^{-1^2} = 3.6788 \, \text{m} \]

Region E

3. \( I_x \) for region D

No integration needed.
Use the formula for the moment of inertia of a rectangle about a centroidal axis.

\[ I_c = \frac{bh^3}{12} \quad (1) \]

So

\[ I_{xD} = \frac{(1 \, \text{m})(2 \times 3.6788 \, \text{m})^3}{12} \]

\[ = 33.1915 \, \text{m}^4 \quad (2) \]
10.1 Moments of Inertia by Integration Example 9, page 3 of 3

5. \( I_x \) for region E. Because the region is symmetrical about the x axis, we can save work by using vertical rectangular strips and by applying Eq. 1 to these strips:

\[
I_c = \frac{1}{12}bh^3
\]

or,

\[
dI_{xE} = \frac{1}{12}(dx)(2y)^3
\]

\[
= \frac{1}{12}(dx)[2(10e^{-x^2})]^3
\]

Thus, with the aid of the integral function on a calculator, we have

\[
I_{xE} = \int dI_x
\]

\[
= \int_1^{1.5} \frac{2000}{3} e^{-3x^2} \, dx
\]

\[
= \frac{47985}{3} \text{ m}^4 \tag{3}
\]

Adding the results for regions D and E gives

\[
I_x = I_{xD} + I_{xE}
\]

\[
\text{by Eq. 2} \quad \text{by Eq. 3}
\]

\[
= 33.1915 + 4.7985
\]

\[
= 38.0 \text{ m}^4 \quad \text{Ans.}
\]
10. Determine the moment of inertia of the crosshatched area about the y axis.
10.1 Moments of Inertia by Integration Example 10, page 2 of 3

1. We want to evaluate

   \[ I_y = \int x^2 \, dA \]  \hspace{1cm} (1)

where \( x \) has the same value throughout the differential element \( dA \).

2. Given the shape of the area, we have to evaluate the integral in Eq. 1 over two sub-regions, B and C.

3. Integrate over region B.

4. \( dA = \) area of rectangle

   \[ = (y_2 - y_1) \times dx \]

   \[ \leq (3x - x^2) \, dx \]

5. \( I_{yB} = \int x^2 \, dA \)

   \[ = \int_0^1 x^2(3x - x^2) \, dx \]

   \[ = 0.5500 \, \text{in}^4 \]  \hspace{1cm} (2)
Integrate over region C.

Write the equation of the line:
\[
\frac{x - 1}{2 - 1} = \frac{y - 3}{4 - 3}
\]
Solving gives
\[
y = x + 2
\]

\[dA = \text{Area of rectangle} = (y_2 - y_1) \times dx\]
\[= [(x + 2) - x^2] \, dx\]

\[I_{yC} = \int x^2 \, dA\]
\[= \int_1^2 x^2( x + 2 - x^2) \, dx\]
\[= 2.2167 \text{ in}^4 \quad (3)\]

Adding the results for regions B and C gives
\[I_y = I_{yB} + I_{yC}\]
\[\text{by Eq. 2} \quad \text{by Eq. 3}\]
\[= 0.5500 + 2.2167 = 2.77 \text{ in}^4 \quad \text{Ans.}\]
10.1 Moments of Inertia by Integration Example 11, page 1 of 2

11. Given that the centroid C of the area bounded by a quarter-circle lies a distance $4a/(3\pi)$ above the base of the quarter-circle, determine the moment of inertia of the cross hatched area about an axis $x_c$ through the centroid.

We want to evaluate

$$I_{xc} = \int y_c^2 \, dA$$

but because the equation of the circle is given in terms of $x$ and $y$ instead of $x_c$ and $y_c$, it is easier to evaluate

$$I_x = \int y^2 \, dA$$

and then use the parallel-axis theorem:

$$I_x = I_{xc} + Ad^2$$

Area of quarter circle

$$= I_{xc} + \frac{\pi a^2}{4} \left(\frac{4a}{3\pi}\right)^2$$

Solving gives

$$I_{xc} = I_x - \frac{4a^4}{9\pi}$$

(1)

Thus now we need to calculate $I_x$. 

\[ x^2 + y^2 = a^2 \]
10.1 Moments of Inertia by Integration Example 11, page 2 of 2

\[ x^2 + y^2 = a^2 \]

\( dA = \text{Area of rectangle} \)
\[ = x \times dy \]

3. Solve \( x^2 + y^2 = a^2 \)
to get
\[ x = \sqrt{a^2 - y^2} \]

4. Substitute into the equation for \( dA \)
\[ dA = x \times dy \]
\[ = \sqrt{a^2 - y^2} \, dy \]

5. Integrate
\[ I_x = \int y^2 \, dA \]
\[ = \int_0^a y^2 \sqrt{a^2 - y^2} \, dy \]
\[ = \frac{\pi a^4}{16} \] (2)

6. Evaluate the integral with a calculator that does symbolic integration or use a table.

7. Use the result given by Eq. 2 in Eq. 1:
\[ I_{xc} = I_x - \frac{4a^4}{9\pi} \]
\[ = \frac{\pi a^4}{16} - \frac{4a^4}{9\pi} = \left( \frac{\pi}{16} - \frac{4}{9\pi} \right) a^4 \]
\[ \leftarrow \text{Ans.} \]