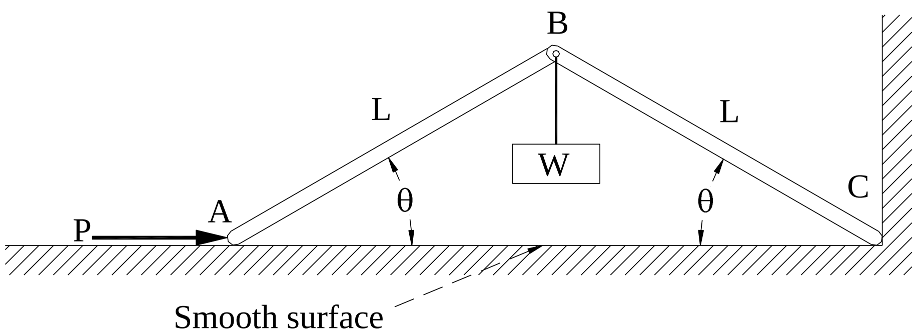
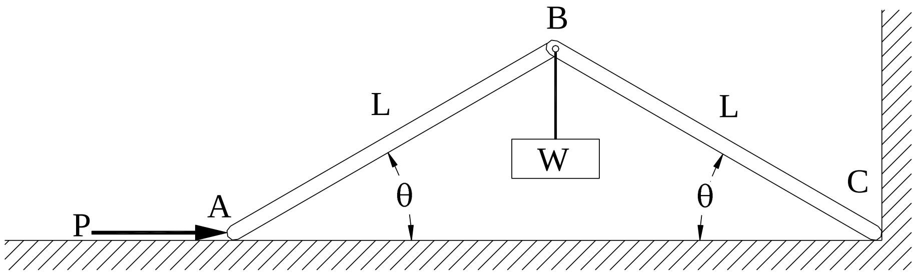
11.1 Virtual Work

11.1 Virtual Work Example 1, page 1 of 5

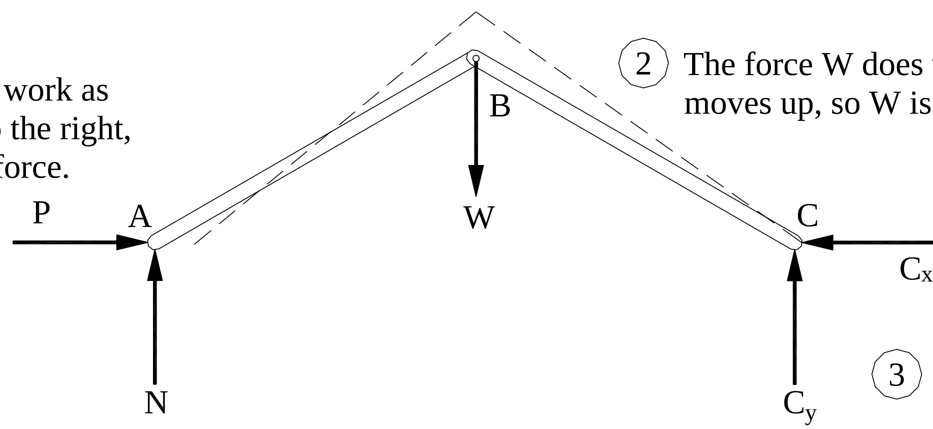
1. Determine the force P required to keep the two rods in equilibrium when the angle $\theta = 30°$ and weight W is 50 lb. The rods are each of length L and of negligible weight. They are prevented from moving out of the plane of the figure by supports not shown.





(1) The system has one degree of freedom, because specifying the value of a single coordinate, θ, completely determines the configuration (shape) of the system. Consider a free-body diagram and identify the active forces—those forces that would do work if θ were increased slightly.

Free-body diagram (The dashed line shows the position of the system after θ has been increased a small amount.)



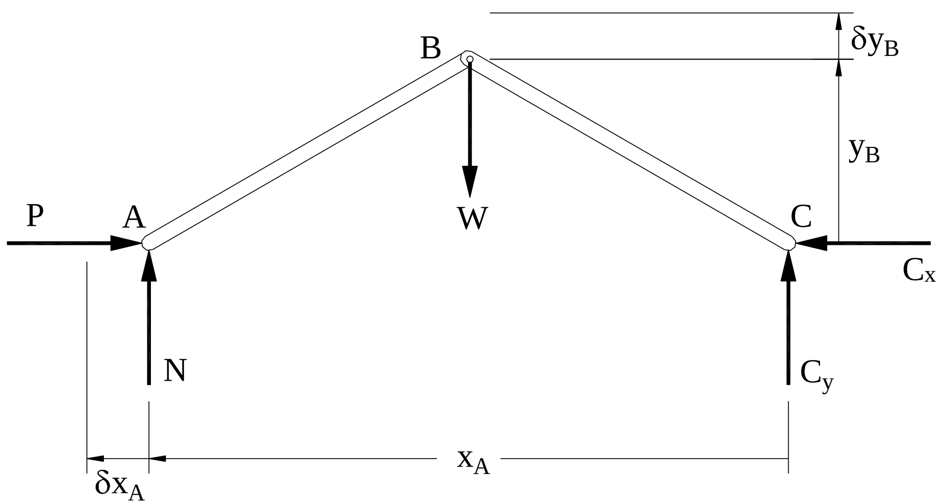
(2) The force W does work because point B moves up, so W is an active force.

(4) The force P does work as point A moves to the right, so P is an active force.

(3) The reactions C_x and C_y do no work because point C does not move. Thus C_x and C_y are not active forces.

(5) The normal force N does no work because it is perpendicular to the displacement of point A. Thus N is not an active force.

6 Introduce coordinates measured from a fixed point, point C in the figure, to the point of application of the active forces.



7 Compute the work done when the coordinates are increased positive infinitesimal amounts, δx_A and δx_B (The custom followed by textbook writers is to use the Greek letter δ rather than simply writing dx_A and dy_B because the infinitesimals represent hypothetical motions—motions that are possible but are not necessarily motions that actually occur). The principle of virtual work says that the total work must add to zero for all possible motions, real or hypothetical—that is, "virtual."

$$\delta U = 0: \quad -P\,\delta x_A - W\,\delta y_B = 0 \qquad (1)$$

A negative sign is present because the force P and displacement δx_A are in opposite directions. That is, P does negative work (absorbs work from the system rather than adding work to the system). Similarly, the force W does negative work because W points down and δy_B is directed up.

8 Relate the differentials δx_A and δy_B through the change in the angle, $\delta\theta$: From the figure, it follows that

$$y_B = L \sin\theta \qquad (2)$$

To relate δy_B to $\delta\theta$, use the ordinary formula from calculus for calculating a differential: if $y = f(\theta)$, then the differential is

$$dy = \frac{df}{d\theta}\, d\theta$$

Applying this formula to Eq. 2 and using δ rather than d gives

$$\delta y_B = L \cos\theta\, \delta\theta \qquad (3)$$

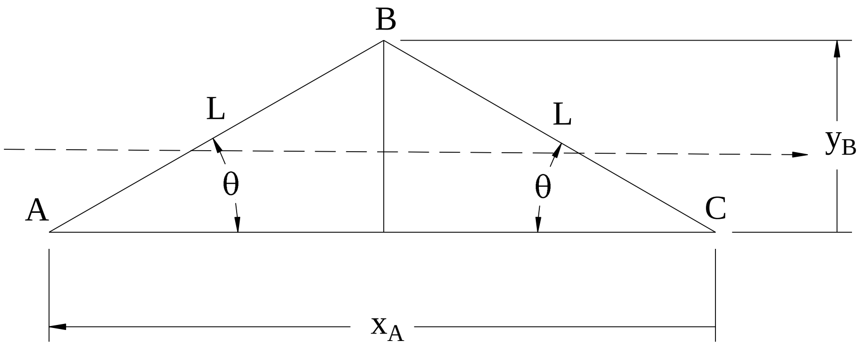
Similarly

$$x_A = 2L \cos\theta$$

$$\delta x_A = -2L \sin\theta\, \delta\theta \qquad (4)$$

Substitute Eqs. 3 and 4 for δy_B and δx_A into the virtual-work equation:

$$-P\,\overbrace{\cancel{\delta x_A}}^{-2L \sin\theta\, \delta\theta} - W\,\underbrace{\cancel{\delta y_B}}_{L \cos\theta\, \delta\theta} = 0 \qquad \text{(Eq. 1 repeated)}$$



9 or,

$$(2P \sin\theta - W \cos\theta)(L\, \delta\theta) = 0$$

Because $L\, \delta\theta \neq 0$, it follows that

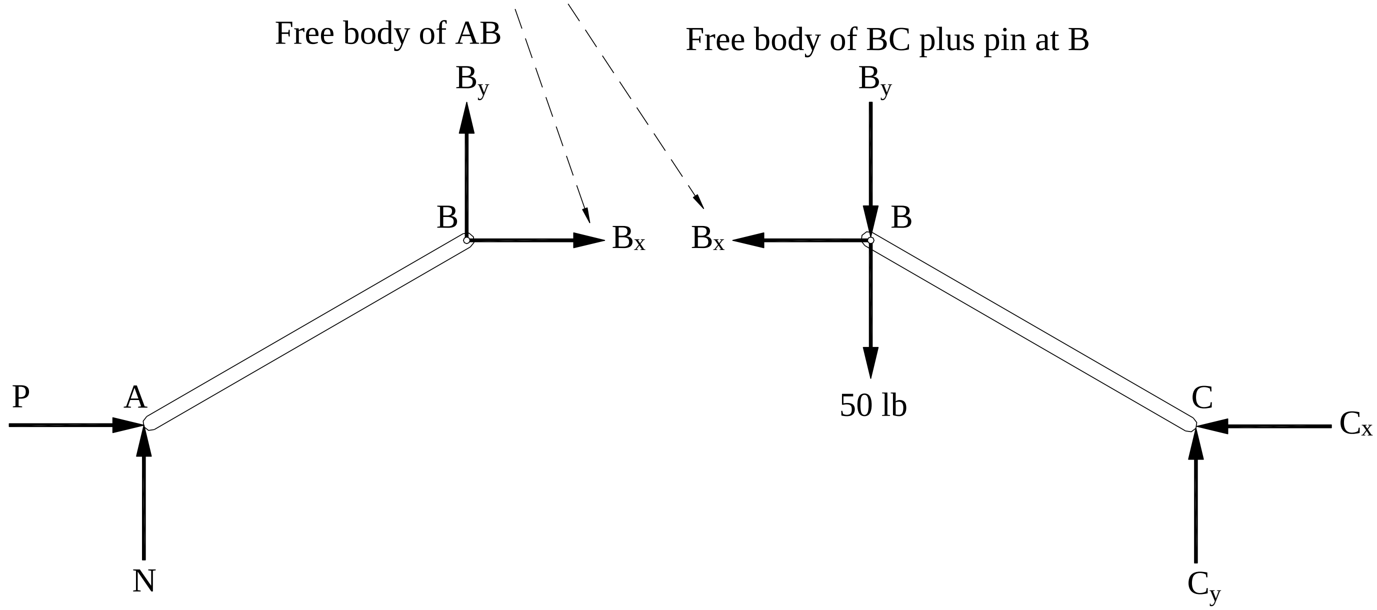
$$2P \sin\theta - W \cos\theta = 0$$

Substituting the given values $\theta = 30°$ and $W = 50$ lb and solving gives

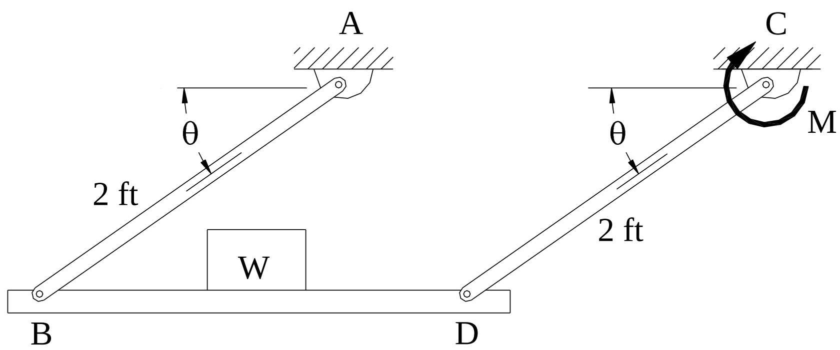
$$P = 43.3 \text{ lb} \qquad \leftarrow \text{Ans.}$$

10 Observation: the forces acting between the rods at pin B never occurred in the virtual-work equation because the work done by the equal-and-opposite force pairs acting between the parts of the body cancel out—for example the work done by B_x acting on rod AB has the opposite sign of the work done by B_x acting on rod BC. Forces such as B_x and B_y would have had to be considered if equilibrium equations rather than virtual work had been used.

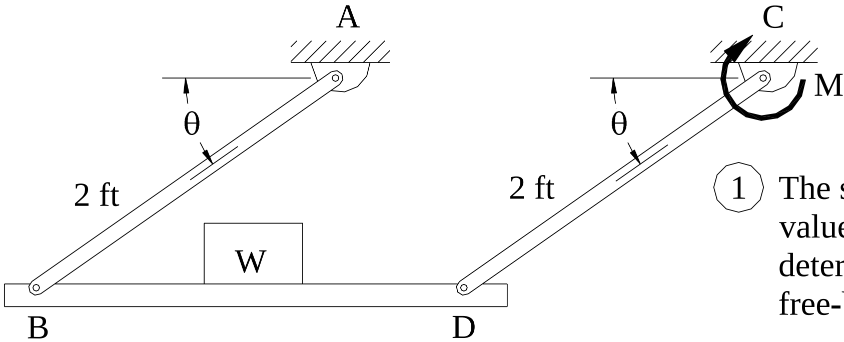
11 Often virtual work is easier to use than equilibrium equations for problems involving connected rigid bodies (typically machines and mechanisms), but this advantage exists *only if the relation between displacements can be found easily*. If the geometry is difficult, then using equilibrium equations is probably the better approach.



2. Determine the value of moment M required to maintain the mechanism in the position shown, if $\theta = 35°$ and W = 200 lb.



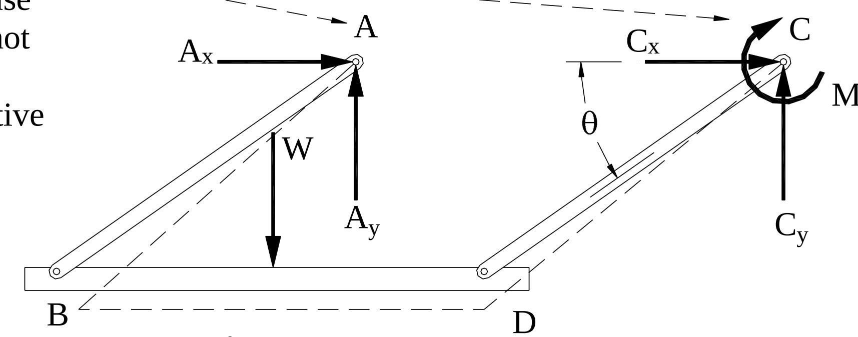
11.1 Virtual Work Example 2, page 2 of 3



1 The system has one degree of freedom: specifying the value of the single coordinate, θ, completely determines the configuration of the system. Consider a free-body diagram and identify the active forces,

Free-body diagram (The dashed line shows the position of the system after θ has been increased a small amount.)

2 The reactions at A and C do no work because points A and C do not move. Thus the reactions are not active forces.

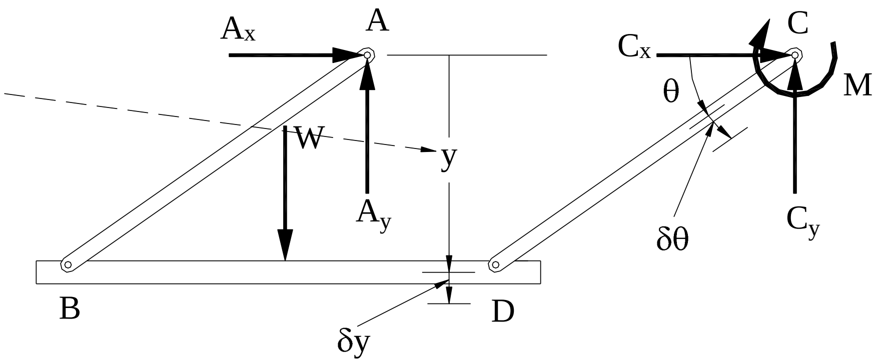


3 Couple-moment M does work because member CD rotates, so M is an active "force" (better said, "an active moment" or "active generalized force")

4 The weight W of the block does work because the center of gravity of the block moves vertically; thus the weight is an active force.

11.1 Virtual Work Example 2, page 3 of 3

5 Introduce a coordinate y measured from the fixed point A to the point of application of the force W.



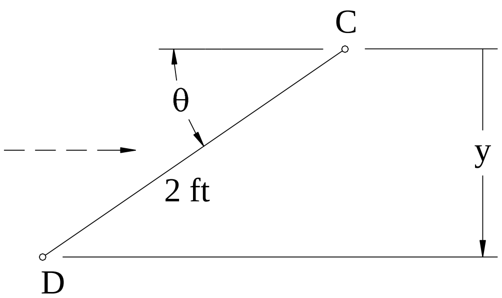
Compute the work done when y and θ are increased a positive infinitesimal amount.

$$\delta U = -M\,\delta\theta + W\,\delta y = 0 \qquad (1)$$

Note that the work done by a moment equals moment times angle of rotation. Here the work is negative because M and δθ have opposite senses.

Next use geometry to relate the y and θ:

$$y = (2\text{ ft})\sin\theta$$



Differentiating gives

$$\delta y = 2\cos\theta\,\delta\theta \qquad (2)$$

Substitute Eq. 2 for δy into Eq. 1.

$$-M\,\delta\theta + W\,\underbrace{\delta y}_{2\cos\theta\,\delta\theta} = 0 \qquad \text{(Eq. 1 repeated)}$$

Thus

$$(-M + 2W\cos\theta)\,\delta\theta = 0 \qquad (3)$$

6 Substituting the given values W = 200 lb and θ = 35° into Eq. 3 and noting δθ ≠ 0 gives

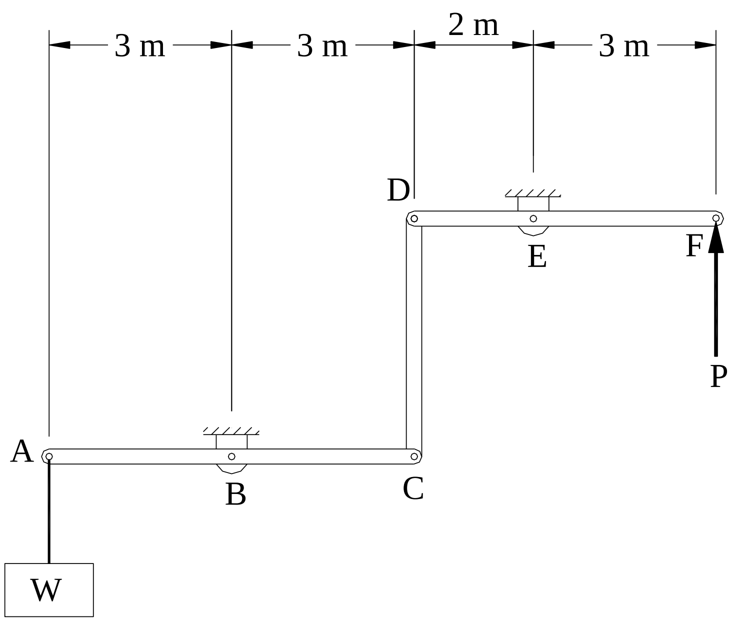
$$-M + 2(200\text{ lb})\cos 35° = 0$$

Solving gives

$$M = 328\text{ lb}\cdot\text{ft} \qquad \leftarrow\text{Ans.}$$

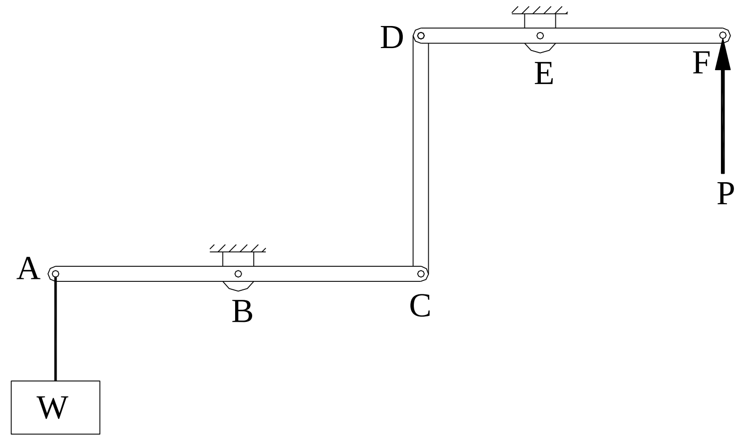
11.1 Virtual Work Example 3, page 1 of 4

3. Determine the value of the weight W required to maintain the mechanism in the position shown, if P = 50 N.

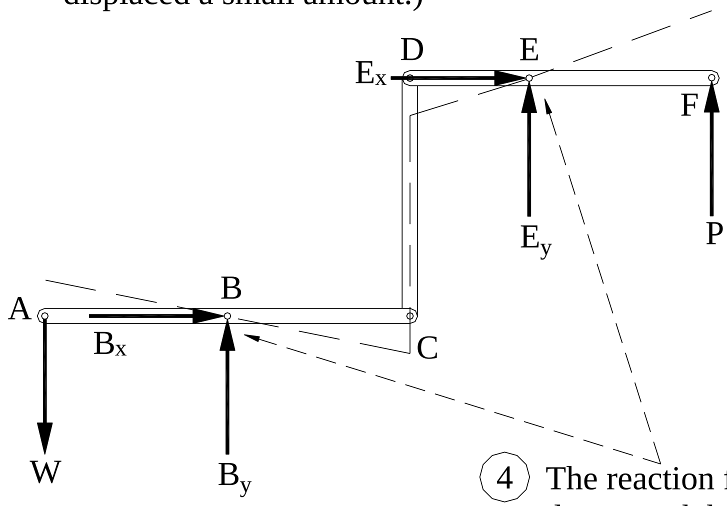


11.1 Virtual Work Example 3, page 2 of 4

1. The system has one degree of freedom because if the displacement of one end of a bar is known, the displacement of the other bars can be found by similar triangles (as will be shown below). Consider a free-body diagram and identify the active forces corresponding to a small change in configuration of the system.



Free-body diagram (The dashed line shows the position of the system after the bars have been displaced a small amount.)

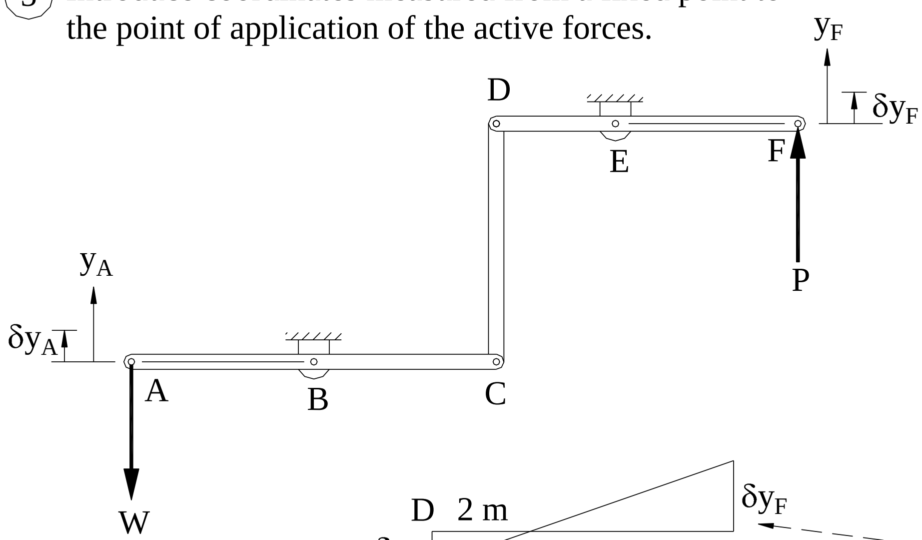


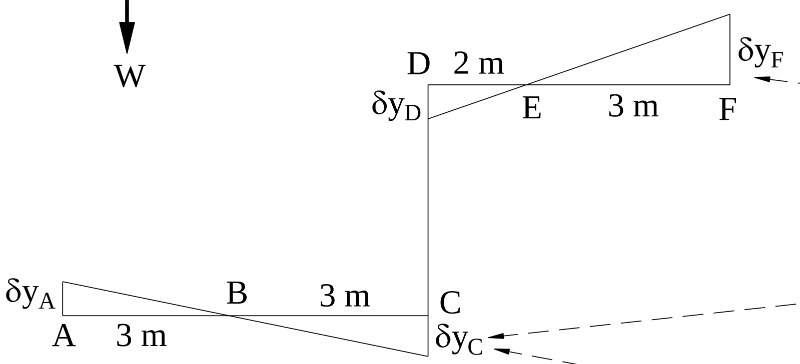
3. The force P does work as point F moves vertically, so P is an active force.

4. The reaction forces at B and E do no work because B and E do not move; thus the reactions are not active forces.

2. The force W does work if A moves vertically, so W is an active force.

5 Introduce coordinates measured from a fixed point to the point of application of the active forces.





6 Compute the work done when the coordinates are increased a positive infinitesimal amount.

$$\delta U = -W\,\delta y_A + P\,\delta y_F = 0 \qquad (1)$$

The force W does negative work because it is directed down, while the displacement is up.

7 Now relate the differentials δy_A and δy_F. By similar triangles

$$\delta y_D/2 = \delta y_F/3 \qquad (2)$$

and

$$\delta y_A/3 = \delta y_C/3 \qquad (3)$$

Member DC does not change length so ends C and D move down the same amount, that is,

$$\delta y_C = \delta y_D \qquad (4)$$

Eqs. 2, 3, and 4 imply

$$\delta y_A = (2/3)\delta y_F \qquad (5)$$

11.1 Virtual Work Example 3, page 4 of 4

8 Substitute Eq. 5 into the virtual-work equation, Eq. 1:

$$-W\,\delta y_A + P\,\delta y_F = 0 \qquad \text{(Eq. 1 repeated)}$$

$$(2/3)\,\delta y_F$$

Thus

$$[-W(2/3) + P]\,\delta y_F = 0$$

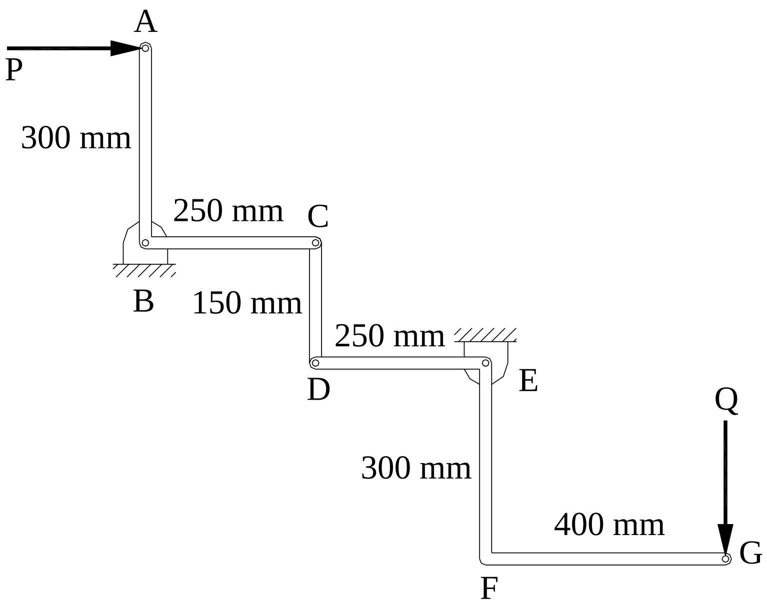
or, since $\delta y_F \neq 0$ and P is given as 50 N,

$$-W(2/3) + 50\text{ N} = 0$$

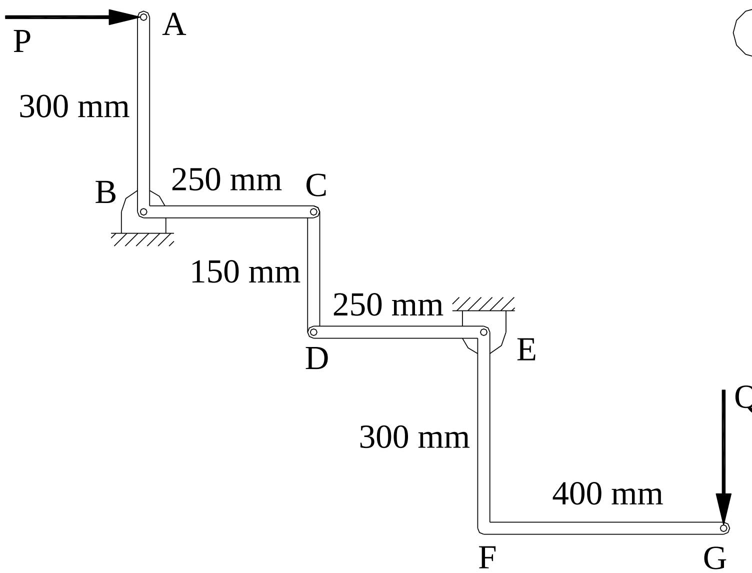
Solving gives

$$W = 75\text{ N} \qquad \leftarrow \text{Ans.}$$

4. Determine the force Q necessary to maintain equilibrium when force P = 400 N.

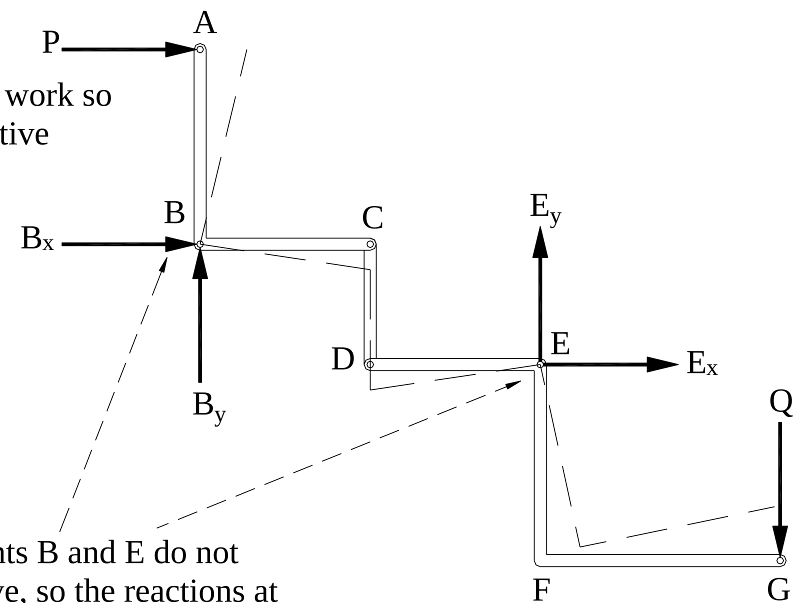


11.1 Virtual Work Example 4, page 2 of 7



1 The system has one degree of freedom because if member ABC is rotated about point B a small amount, then the position of CD and DEFG can be determined. Consider a free-body diagram and identify the active forces corresponding to the displacements shown.

Free-body diagram (The dashed line shows the position of the system after the bars have been displaced a small amount.)

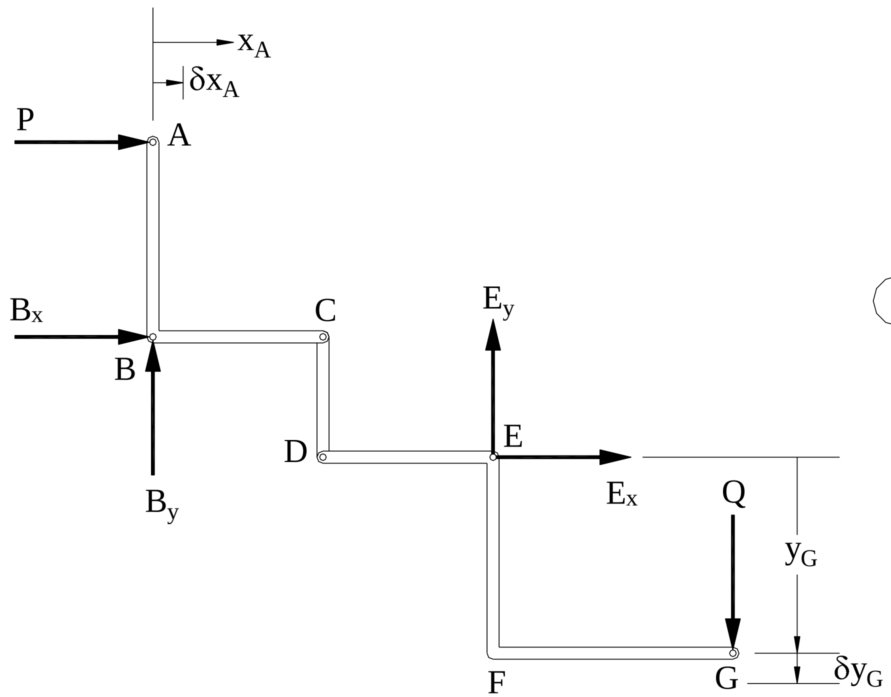


2 P does work so is an active force.

3 Points B and E do not move, so the reactions at these points do no work.

4 Q does work so is an active force.

5 Introduce coordinates measured from a fixed point to the point of application of the forces.

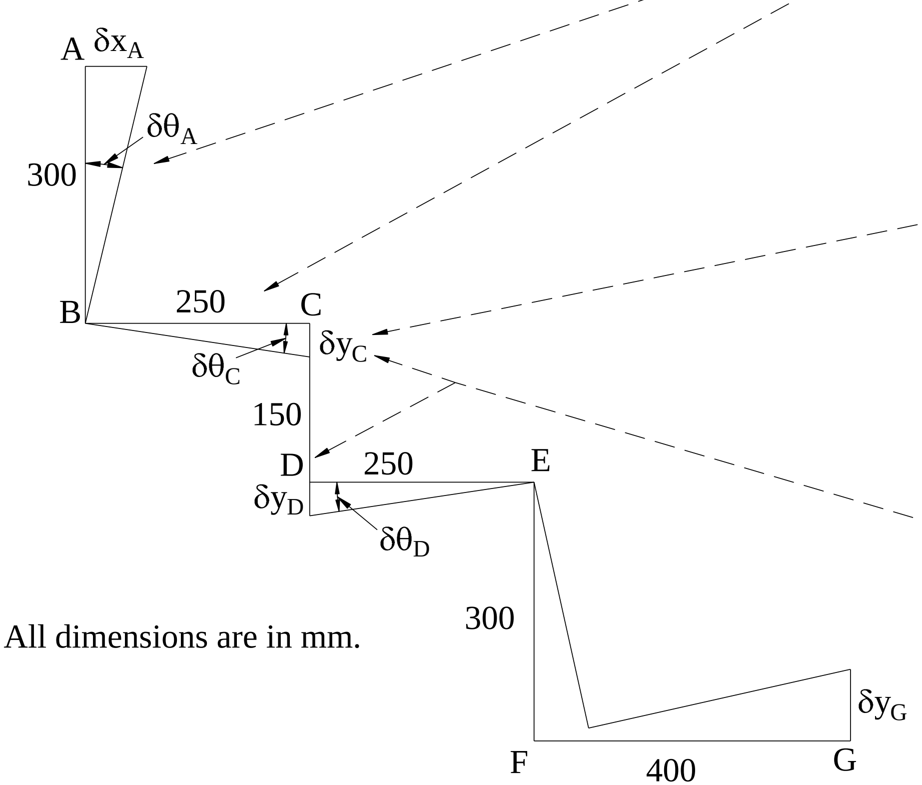


6 Compute the work done when the coordinates are increased a positive infinitesimal amount.

$$\delta U = 0: \; P\,\delta x_A + Q\,\delta y_G = 0 \qquad (1)$$

7 Relate the differentials δx_A and δy_G. Begin by noting that because $\delta\theta_A$ is a small angle, the tangent of $\delta\theta_A$ can be replaced by the angle itself:

$$\delta\theta_A = \frac{\delta x_A}{300 \text{ mm}} \qquad (2)$$



All dimensions are in mm.

8 Member ABC is a rigid body, and all parts must rotate the same amount. Thus

$$\delta\theta_C = \delta\theta_A$$

Substituting for $\delta\theta_A$ from Eq. 2 then gives

$$\delta\theta_C = \frac{\delta x_A}{300 \text{ mm}}$$

9 Again using the small angle approximation for the tangent gives

$$\delta y_C = (250 \text{ mm})\, \delta\theta_C$$

$$= (250 \text{ mm})\left(\frac{\delta x_A}{300 \text{ mm}}\right)$$

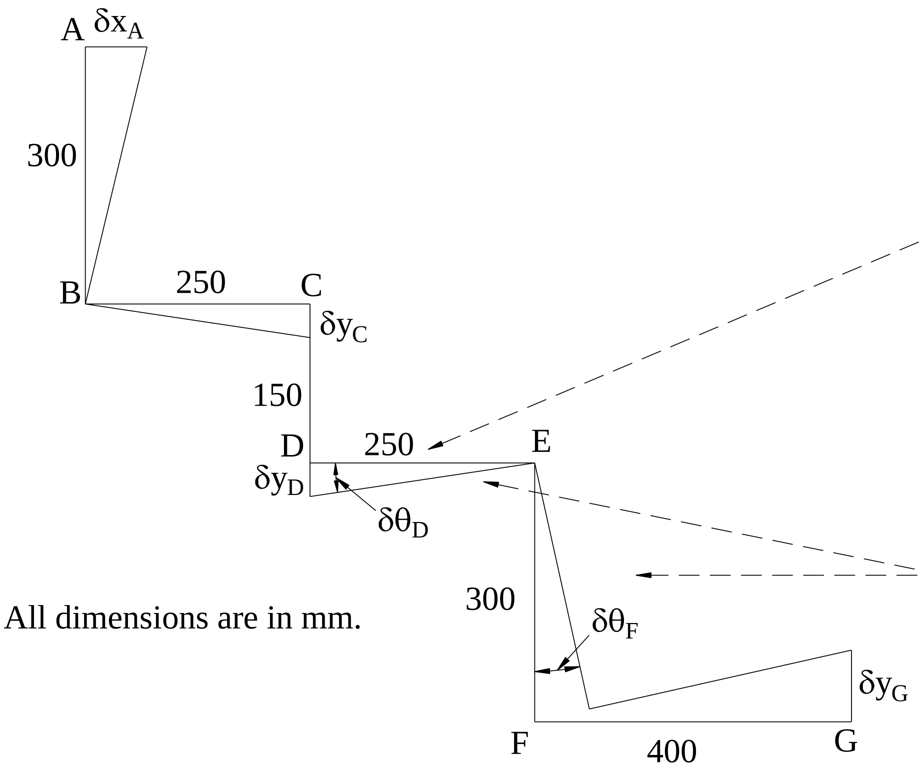
$$= (5/6)\, \delta x_A$$

10 Member CD is a rigid body and thus doesn't shorten or lengthen. It follows that

$$\delta y_D = \delta y_C$$

Thus

$$\delta y_D = (5/6)\, \delta x_A \qquad (3)$$



All dimensions are in mm.

(11) Using the small angle approximation for the tangent gives

$$\delta\theta_D = \frac{\delta y_D}{250\text{ mm}}$$

[(5/6) δx_A], by Eq. 3

$$= \left(\frac{\delta x_A}{300\text{ mm}}\right)$$

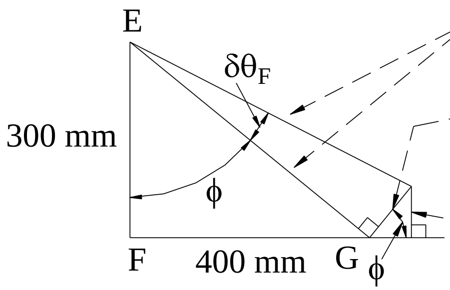
Member DEFG is a rigid body and so all parts must rotate the same amount. Thus

$$\delta\theta_F = \delta\theta_D$$

(12) Thus

$$\delta\theta_F = \left(\frac{\delta x_A}{300\text{ mm}}\right) \qquad (4)$$

13 The remaining step is to relate δy_G to $\delta\theta_F$. We can do this in two ways, by geometry or by calculus. Let's begin with the geometric approach. First consider a rotation of line EG by an amount $\delta\theta_F$.



14 Length = $\sqrt{(300\text{ mm})^2 + (400\text{ mm})^2}$ = 500 mm

15 (500 mm) $\delta\theta_F$

16 δy_G is the vertical distance that point G moves up. So, considering the small triangle gives

$$\frac{400\text{ mm}}{500\text{ mm}} = \frac{4}{5}$$

$$|\delta y_G| = (500\ \delta\theta_F)\sin\phi$$

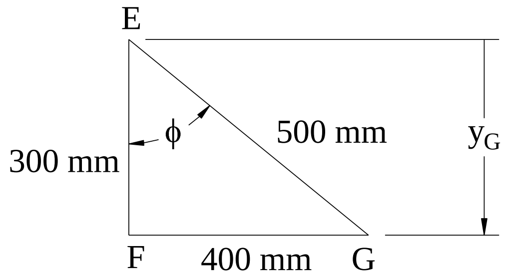
$$\frac{\delta x_A}{300}\text{, by Eq. 4}$$

or,

$$\delta y_G = -\frac{4}{3}\,\delta x_A \qquad (5)$$

(Insert a minus sign, because δy_G was originally defined as positive down)

17 Consider an alternative solution for δy_G, based on calculus:



18 $y_G = (500 \text{ mm}) \cos \phi$

$$\delta y_G = -500 \sin \phi \, \delta\phi$$

$\delta\phi = \delta\theta_F$ because both angles measure the rotation of line EG)

$$= -500\left(\frac{400 \text{ mm}}{500 \text{ mm}}\right) \delta\theta_F$$

$\dfrac{\delta x_A}{300}$, by Eq. 4

$$= -\frac{400 \, \delta x_A}{300}$$

$$= -\frac{4 \, \delta x_A}{3}$$

(Same as Eq. 5)

19 Note that we can't write

$$y_G = 300 \text{ mm}$$

and then differentiate to get δy_G (which would give $\delta y_G = 0$). The equation for y_G must define a continuous and differentiable function, not a relationship that is only valid at a single value of ϕ.

20 Substitute for δy_G from Eq. 5 into the virtual work equation:

$$P \, \delta x_A + Q \, \delta y_G = 0 \qquad \text{(Eq. 1 repeated)}$$

$\dfrac{-4 \, \delta x_A}{3}$, by Eq. 5

or,

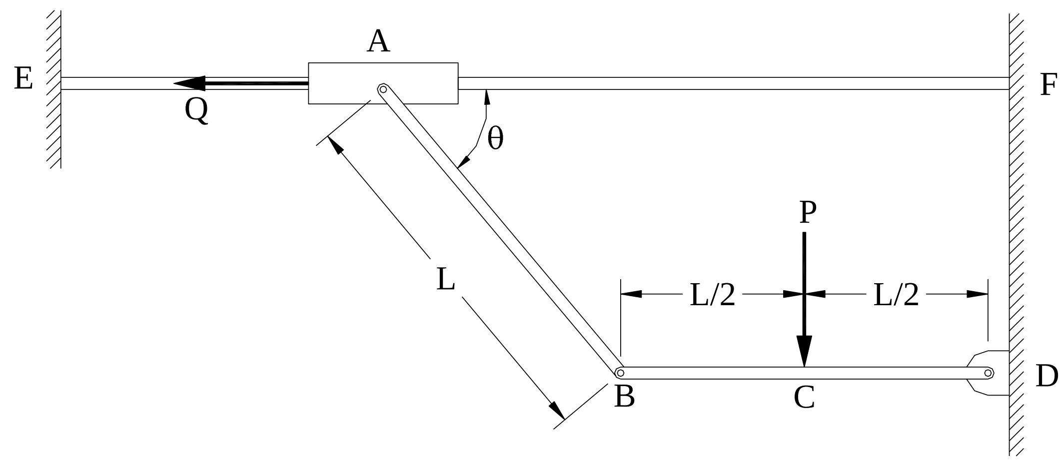
$$[P + (-4/3)Q] \, \delta x_A = 0$$

Dividing through by δx_A and using the given value P = 400 N yields

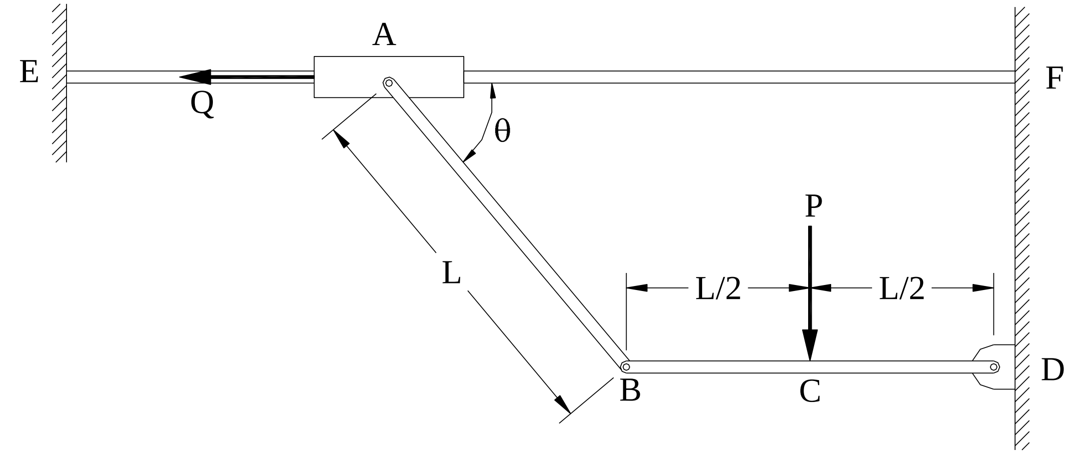
$$Q = 300 \text{ N} \qquad \leftarrow \text{Ans.}$$

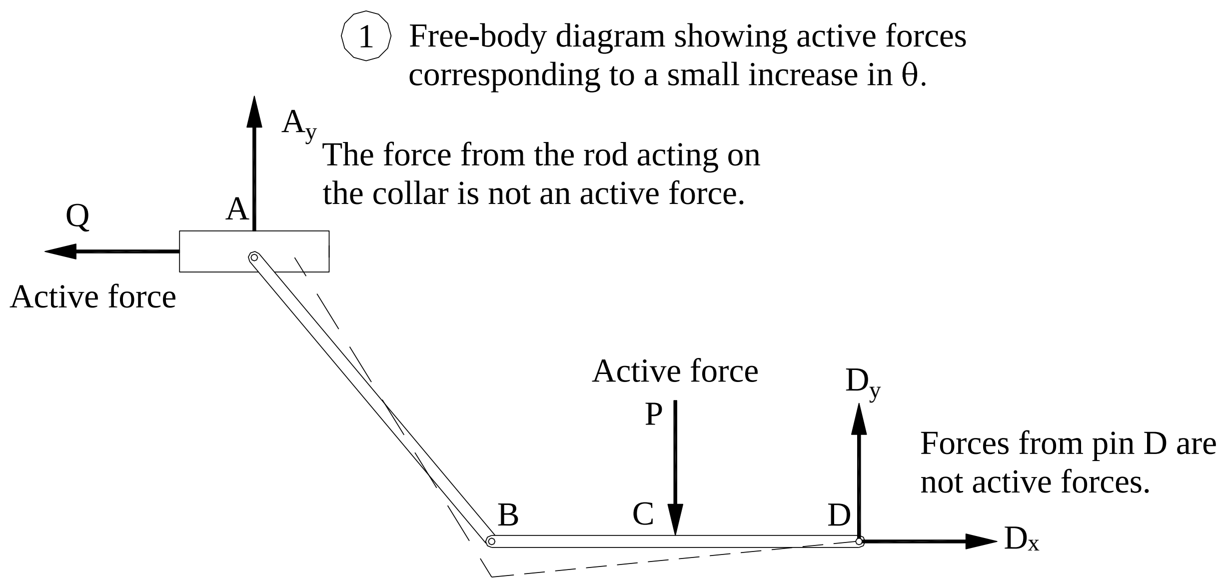
11.1 Virtual Work Example 5, page 1 of 4

5. Link AB is connected to collar A, which can slide with negligible friction on horizontal rod EF. Determine the value of force Q necessary to maintain equilibrium when $\theta = 50°$, L = 300 mm, and P = 100 N.



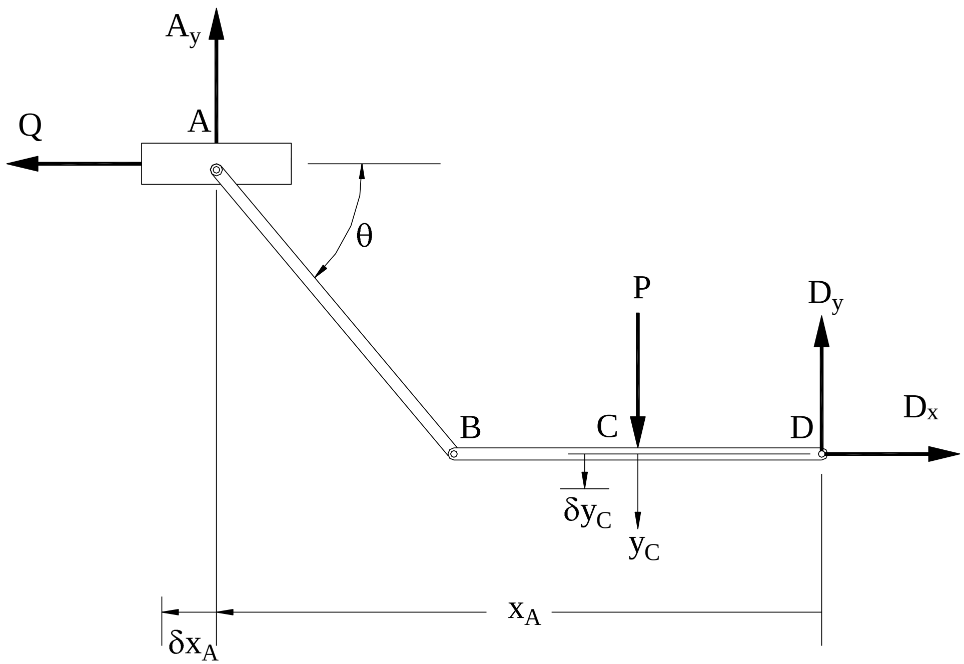
11.1 Virtual Work Example 5, page 2 of 4





11.1 Virtual Work Example 5, page 3 of 4

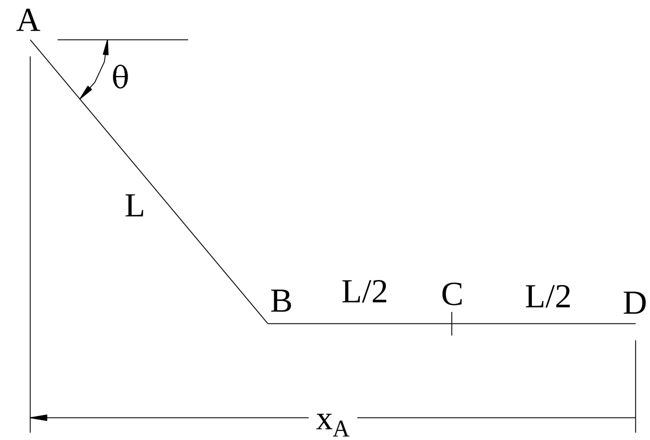
(2) Introduce coordinates measured from the fixed point D to the point of application of the active forces.



(3) Compute the work done when the coordinates are increased a positive infinitesimal amount.

$$\delta U = 0\text{:}\;\; Q\,\delta x_A + P\,\delta y_C = 0 \qquad (1)$$

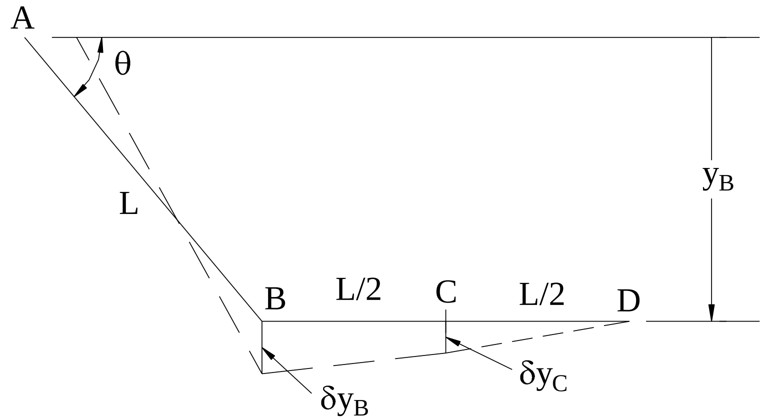
(4) Relate the differential δx_A to the change in angle, $\delta\theta$.



(5) $$x_A = L\cos\theta + \frac{L}{2} + \frac{L}{2}$$

$$\delta x_A = -L\sin\theta\,\delta\theta + \underbrace{\cancel{\frac{\delta L}{2}}}_{0} + \underbrace{\cancel{\frac{\delta L}{2}}}_{0} \qquad (2)$$

(Length L does not change)



6 Relate the differential δy_C to the change in angle $\delta\theta$. From the figure, we have

$$y_B = L \sin\theta$$

$$\delta y_B = L \cos\theta\ \delta\theta$$

By similar triangles,

$$\frac{\cancel{\delta y_B}^{\,L \cos\theta\ \delta\theta}}{L/2 + L/2} = \frac{\delta y_C}{L/2}$$

Thus

$$\delta y_C = \frac{L \cos\theta\ \delta\theta}{2} \qquad (3)$$

7 Substitute Eqs. 2 and 3 for δx_A and δy_C into the virtual work equation, Eq. 1:

$-L \sin\theta\ \delta\theta$, by Eq. 2

$$Q\ \cancel{\delta x_A} + P\ \cancel{\delta y_C} = 0 \qquad \text{(Eq. 1 repeated)}$$

$\dfrac{L \cos\theta\ \delta\theta}{2}$, by Eq. 3

Thus

$$\left(-Q \sin\theta + P\,\frac{\cos\theta}{2}\right)(L\ \delta\theta) = 0$$

Because $L\ \delta\theta \neq 0$, it follow that

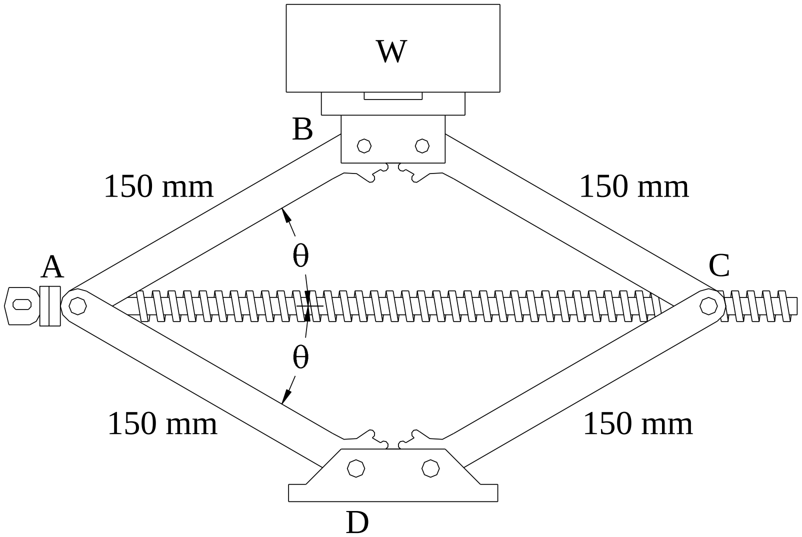
$$-Q \sin\theta + P\,\frac{\cos\theta}{2} = 0$$

Substituting $\theta = 50°$ and $P = 100$ N and solving for Q gives

$$Q = 42.0 \text{ N} \qquad \leftarrow \text{Ans.}$$

11.1 Virtual Work Example 6, page 1 of 5

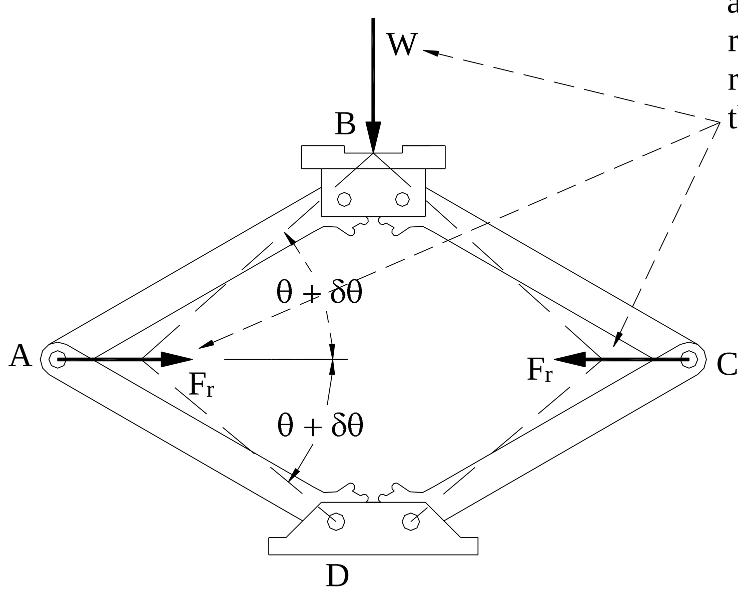
6. Rotating the threaded rod AC of the automobile jack causes joints A and C to move closer together, thus raising the weight W. Determine the axial force in the rod, if θ = 30° and W = 2 kN.



11.1 Virtual Work Example 6, page 2 of 5

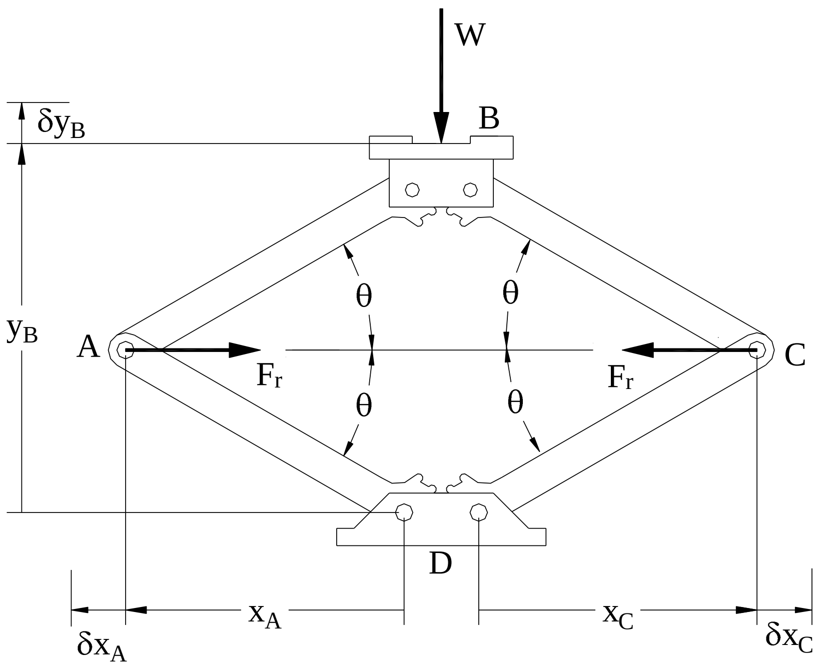
1 The system has one degree of freedom because once θ is specified, the location of all parts of the jack can be determined. Consider a free-body diagram of the jack and identify the active forces corresponding to a small change in θ.

Free-body diagram



2 To get a virtual-work equation that contains the axial force in the rod, it is necessary to *exclude* the rod from the free-body diagram. The effect of the rod is then represented by the two forces F_r. W and the two F_r forces are the active forces.

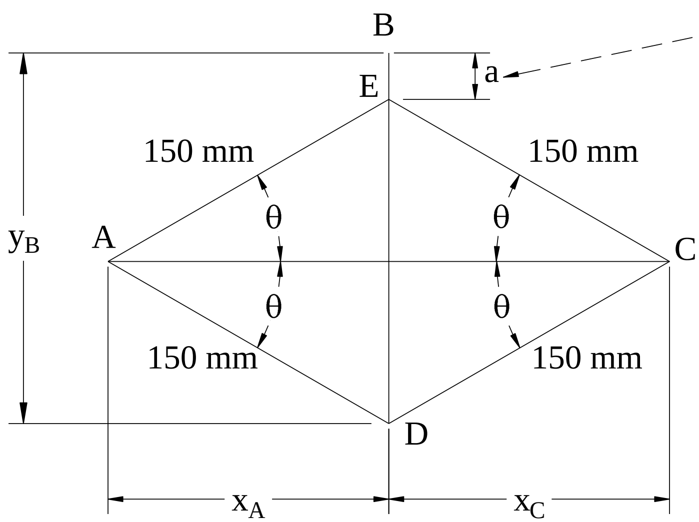
3 Introduce coordinates measured from fixed points to the points of application of the active forces.



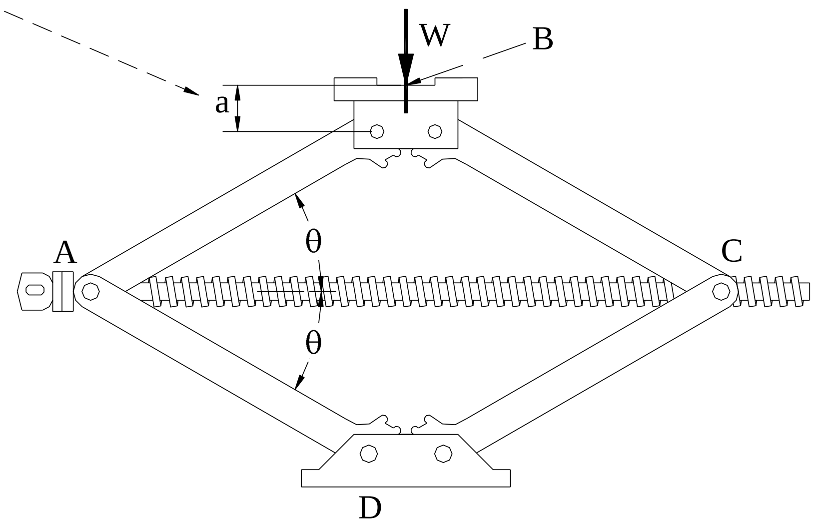
4 Calculate the work done.

$$\delta U = 0: \; -W\,\delta y_B - F_r\,\delta x_A - F_r\,\delta x_C = 0 \tag{1}$$

5 Relate δy_B, δx_A, and δx_C through the angle change, $d\theta$.



6 Distance "a," from the intersection of the two sloping members, point E, to point B, does not change as θ changes. Thus when "a" is differentiated, the result is zero: $\delta a = 0$.



7 $x_A = (150 \text{ mm}) \cos\theta$

$$\delta x_A = -150 \sin\theta \, \delta\theta \qquad (2)$$

$x_C = (150 \text{ mm}) \cos\theta$

$$\delta x_C = -150 \sin\theta \, \delta\theta \qquad (3)$$

$y_B = 2(150 \text{ mm}) \sin\theta + a$

$$\delta y_B = 300 \cos\theta \, \delta\theta + \cancelto{0}{\delta a} \qquad (4)$$

11.1 Virtual Work Example 6, page 5 of 5

8 Substituting the expressions for δy_B, δx_A, and δx_C and into the virtual work equation, Eq. 1, gives

$$-W \overbrace{\delta y_B}^{300\cos\theta\,\delta\theta\text{, by Eq. 4}} - F_r \underbrace{\delta x_A}_{-150\sin\theta\,\delta\theta\text{, by Eq. 2}} - F_r \overbrace{\delta x_C}^{-150\sin\theta\,\delta\theta\text{, by Eq. 3}} = 0 \qquad \text{(Eq. 1 repeated)}$$

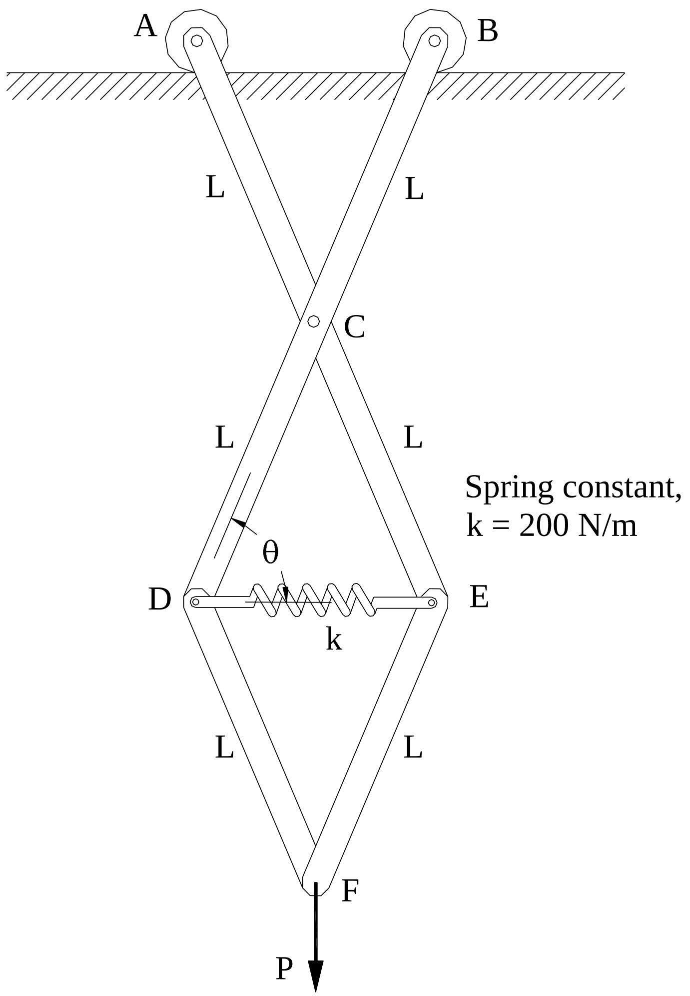
or,

$$[-300W\cos\theta + 2(150)F_r\sin\theta]\,\delta\theta = 0$$

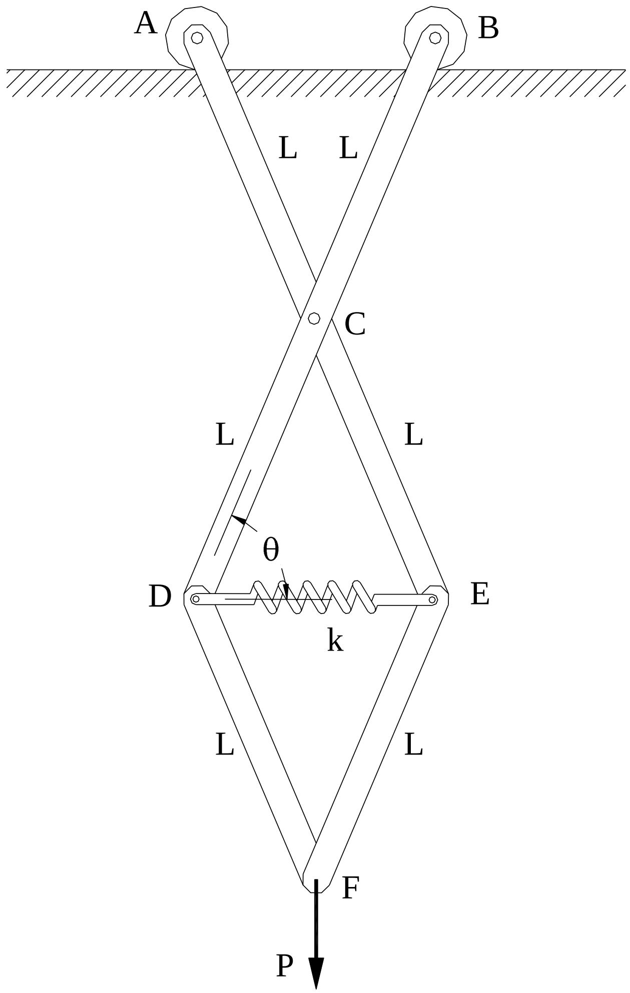
Dividing through by $\delta\theta$, substituting the given values $W = 2$ kN and $\theta = 30°$, and solving gives

$$F_r = 3.46\text{ kN} \qquad \leftarrow\text{Ans.}$$

7. The original length of the spring is L. Determine the angle θ for equilibrium if L = 3 m and P = 300 N.

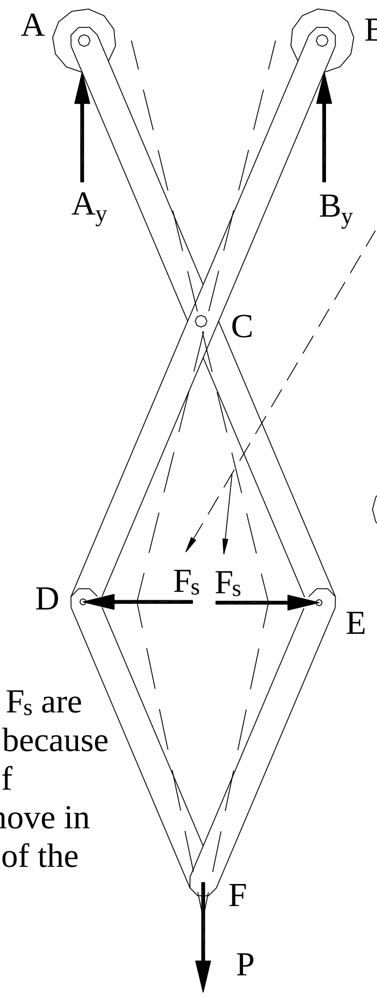


11.1 Virtual Work Example 7, page 2 of 5



(1) The system can be described by a single coordinate, θ. Consider a free-body diagram and identify the active forces corresponding to a small change in θ.

Free-body diagram



(2) Reactions A_y and B_y do no work because they are perpendicular to the displacement of points A and B.

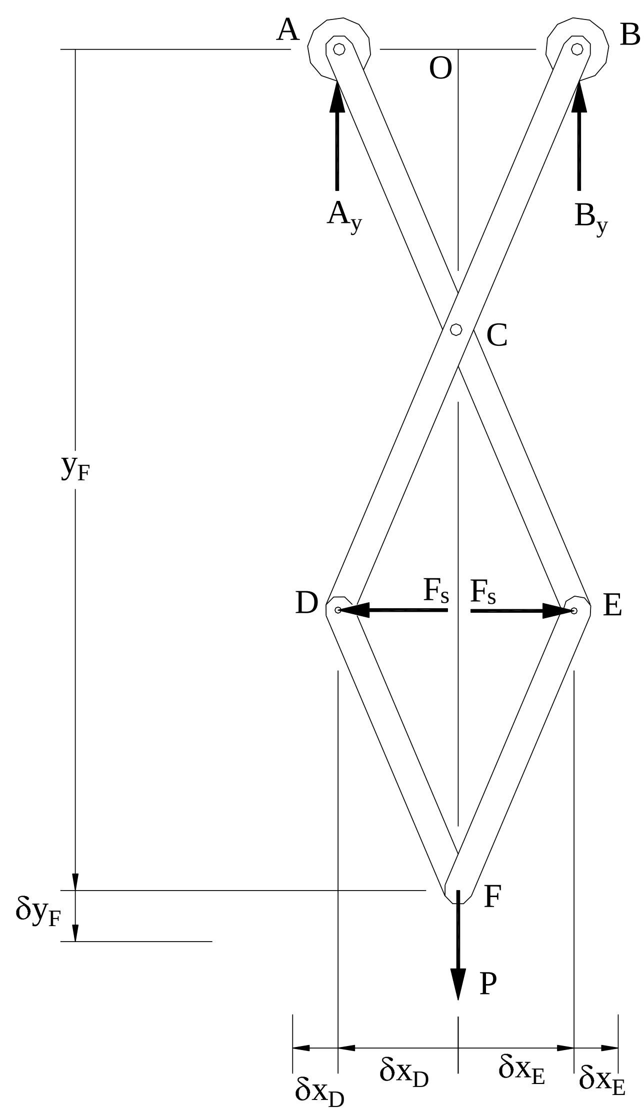
(3) The spring is *not* part of the free-body; the effect of the spring is represented by the forces F_s.

(4) Forces P and F_s are active forces because their points of application move in the direction of the forces.

5 Introduce coordinates measured from the point O directly above pin C to the point of application of the active forces.

6 Compute the work done when the coordinates are increased a positive infinitesimal amount.

$$\delta U = P\,\delta y_F + F_s\,\delta x_D + F_s\,\delta x_E = 0 \qquad (1)$$



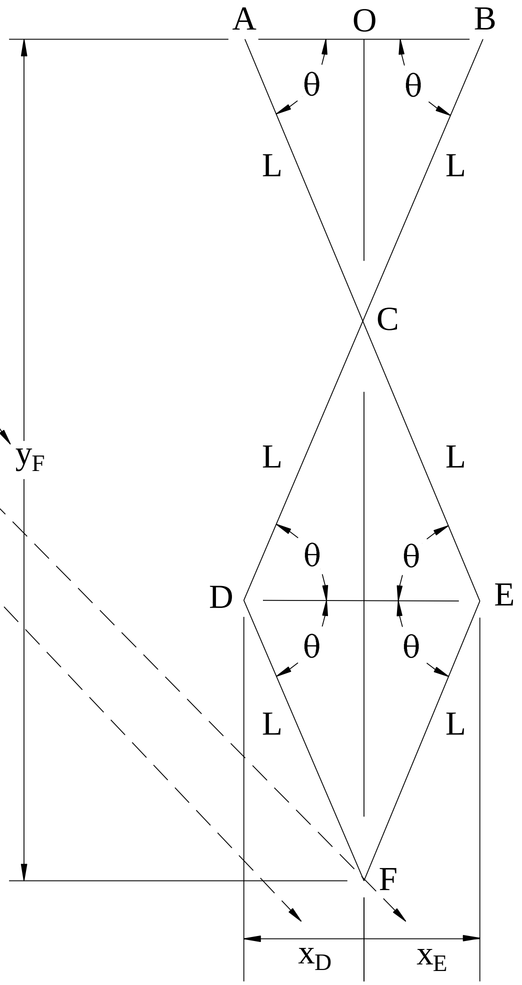
11.1 Virtual Work Example 7, page 4 of 5

7 Relate the differentials δy_F, δx_D, and δx_E to the angle change $\delta\theta$. From the figure, we see that

$$y_F = 3L \sin\theta$$

$$x_E = L \cos\theta$$

$$x_D = L \cos\theta$$



Differentiating gives

$$\delta y_F = 3L \cos\theta\, \delta\theta \quad (2)$$

$$\delta x_E = -L \sin\theta\, \delta\theta \quad (3)$$

$$\delta x_D = -L \sin\theta\, \delta\theta \quad (4)$$

We can use the same figure to calculate the length of the spring, L′, say:

$$L' = \text{distance DE}$$

$$= 2L \cos\theta \quad (5)$$

11.1 Virtual Work Example 7, page 5 of 5

8 The force in the spring is, then,

$$F_s = k \times \text{compression of spring}$$

$$= k \times (\underbrace{\text{original length}}_{L\text{ (given)}} - \underbrace{\text{final length}}_{L' = 2L\cos\theta,\text{ by Eq. 5}})$$

$$= kL(1 - 2\cos\theta) \qquad (6)$$

Substitute from Eqs. 2, 3, 4, and 6 into the virtual work equation, Eq. 1:

$$P\,\underbrace{\delta y_F}_{3L\cos\theta\,\delta\theta\text{ by Eq. 2}} + \overbrace{F_s}^{kL(1-2\cos\theta),\text{ by Eq. 6}}\,\underbrace{\delta x_D}_{-L\sin\theta\,\delta\theta\text{ by Eq. 3}} + F_s\,\underbrace{\delta x_E}_{-L\sin\theta\,\delta\theta\text{ by Eq. 4}} = 0 \qquad \text{(Eq. 1 repeated)}$$

Thus

$$[3P\cos\theta - 2kL(1 - 2\cos\theta)\sin\theta](L\,\delta\theta) = 0$$

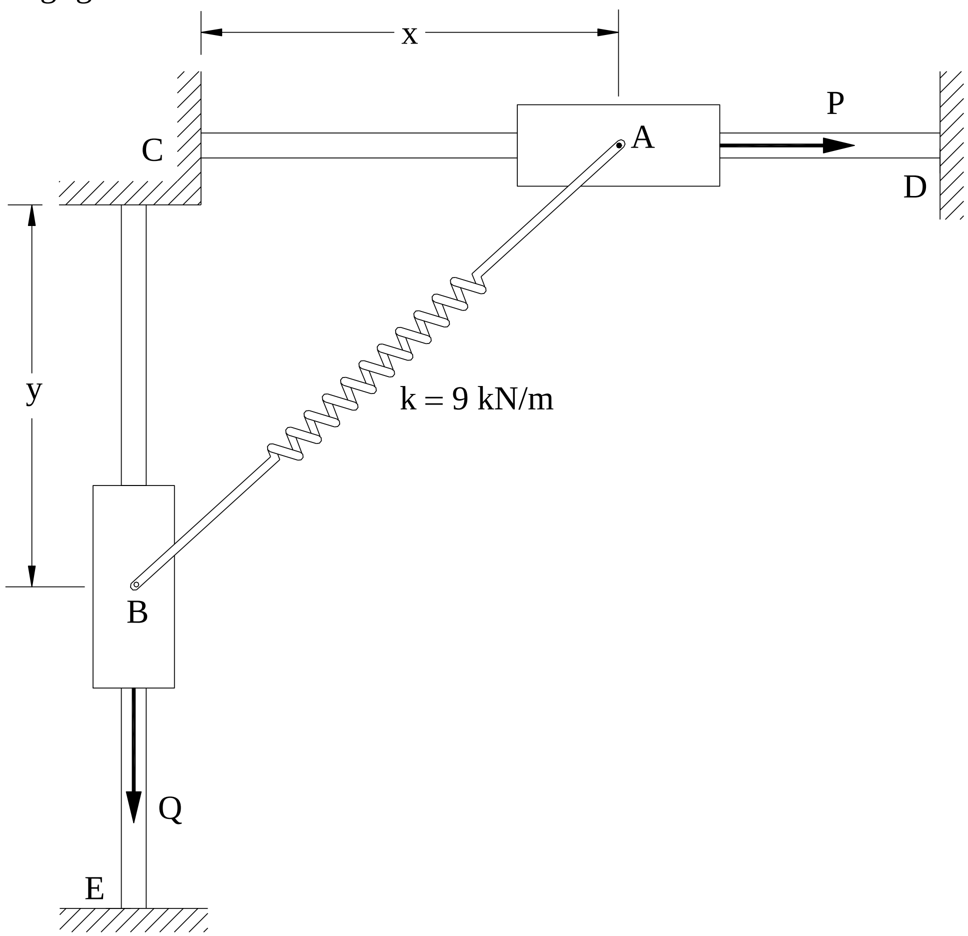
or, since $L\,\delta\theta \neq 0$,

$$[3P\cos\theta - 2kL(1 - 2\cos\theta)\sin\theta] = 0 \qquad (7)$$

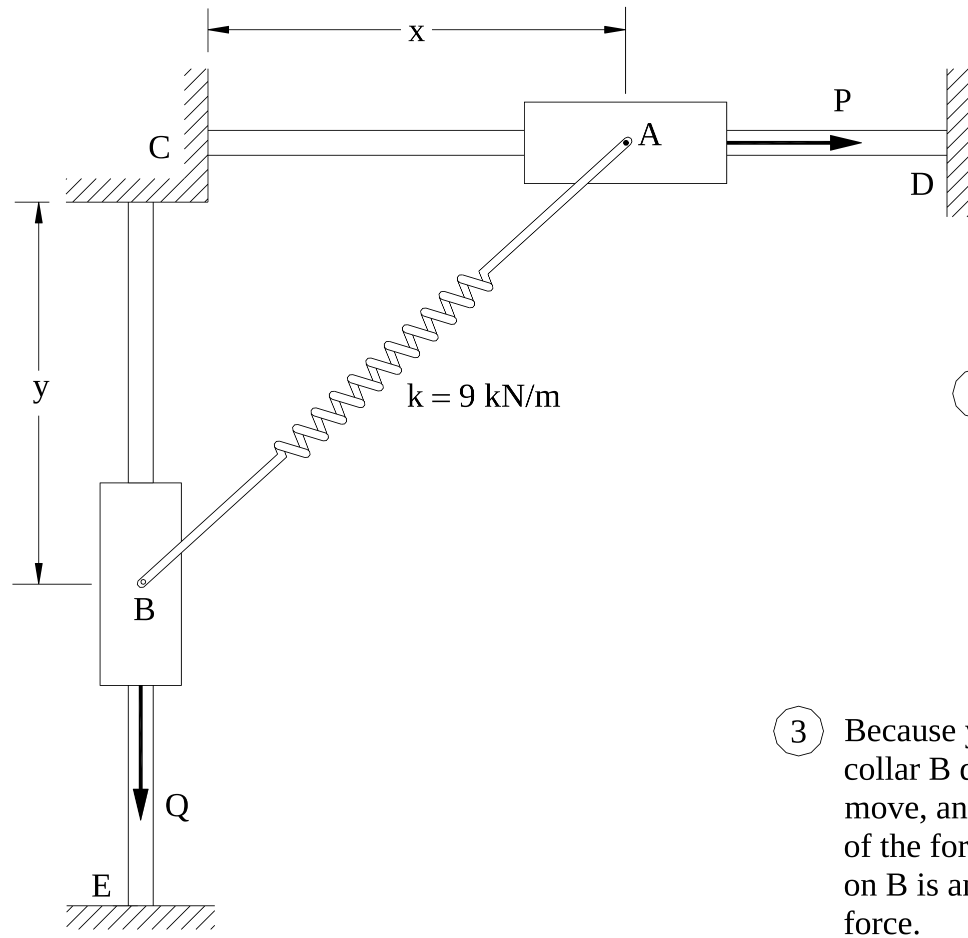
9 Substituting the given values P = 300 N, k = 200 N/m, and L = 3 m into Eq. 7 and solving numerically gives

$$\theta = 69.1^\circ \qquad \leftarrow \text{Ans.}$$

8. Collars A and B can slide freely on rods CD and CE. Determine the values of x and y, given that forces P = 900 N and Q = 800 N. The unstretched length of the spring is 0.2 m, and the weight of the collars is negligible.



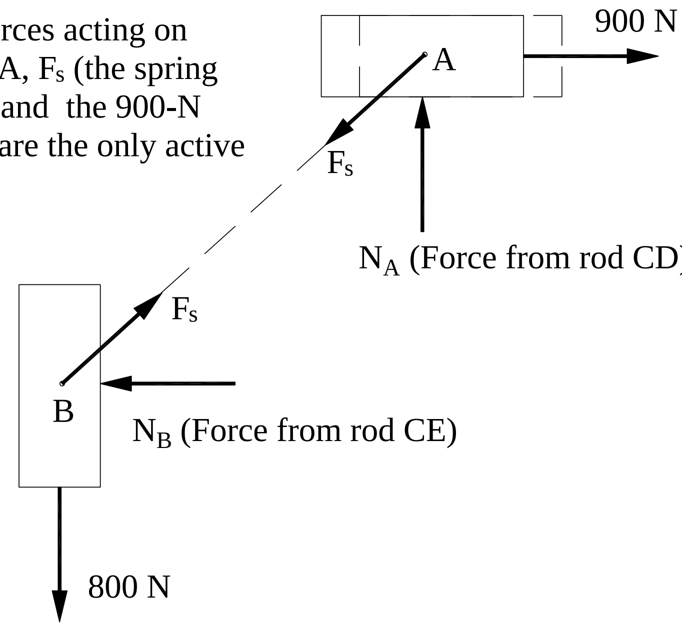
11.1 Virtual Work Example 8, page 2 of 5



1 The system has two degree of freedom since both x and y coordinates must be known if the configuration of the system is to be determined. Consider a free-body diagram and identify the active forces corresponding to a small change in x, while y is held fixed.

Free-body diagram

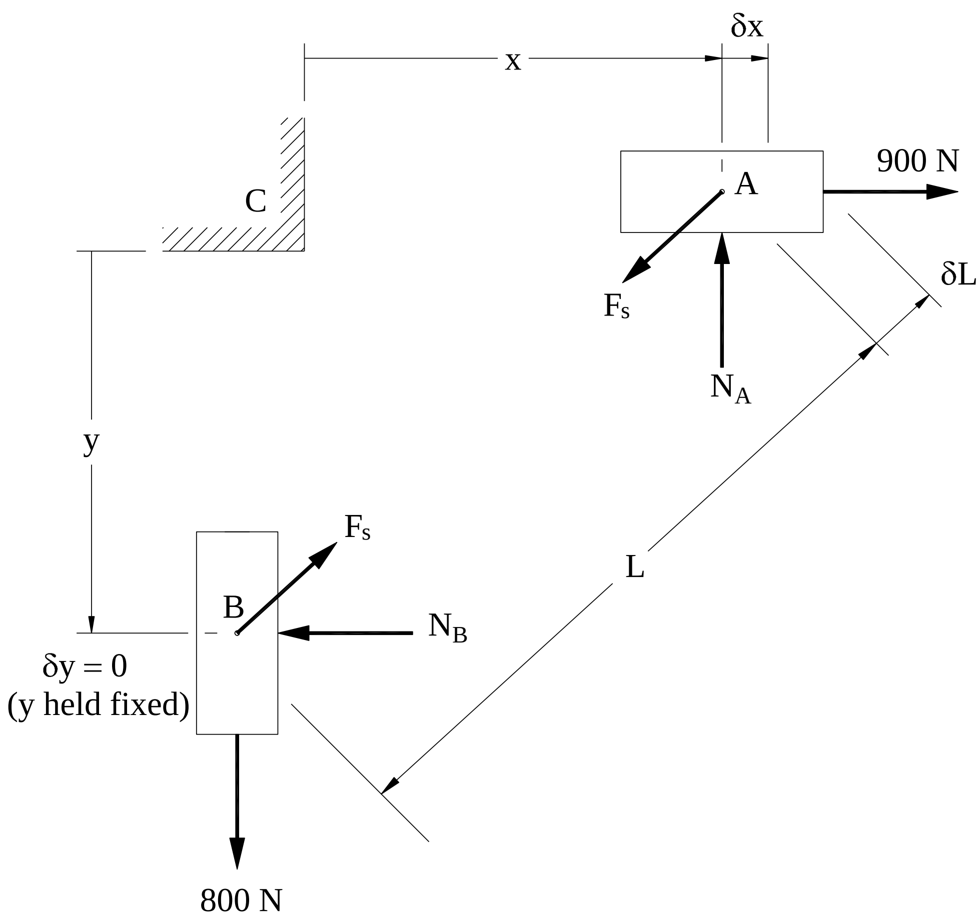
2 The forces acting on collar A, F_s (the spring force) and the 900-N force, are the only active forces.



3 Because y is fixed, collar B does not move, and so none of the forces acting on B is an active force.

11.1 Virtual Work Example 8, page 3 of 5

4 The coordinate x locates the position of the 700-N force. Introduce an additional coordinate, L, that locates the point of application of the spring force, F_s.



5 Compute the work done:

$$\delta U = 0\text{: } (900 \text{ N})\,\delta x - F_s\,\delta L = 0 \qquad (1)$$

Relate δx and δL:

$$L^2 = x^2 + y^2 \qquad (2)$$

Differentiating gives

$$2L\,\delta L = 2x\,\delta x + 2y\,\cancel{\delta y}$$

0 (Because y is fixed)

Thus

$$\delta L = x/L\,\delta x$$

Introduce the latter equation into Eq. 1:

$$(900 \text{ N})\,\delta x - F_s\,\cancel{\delta L} = 0 \qquad \text{(Eq. 1 repeated)}$$

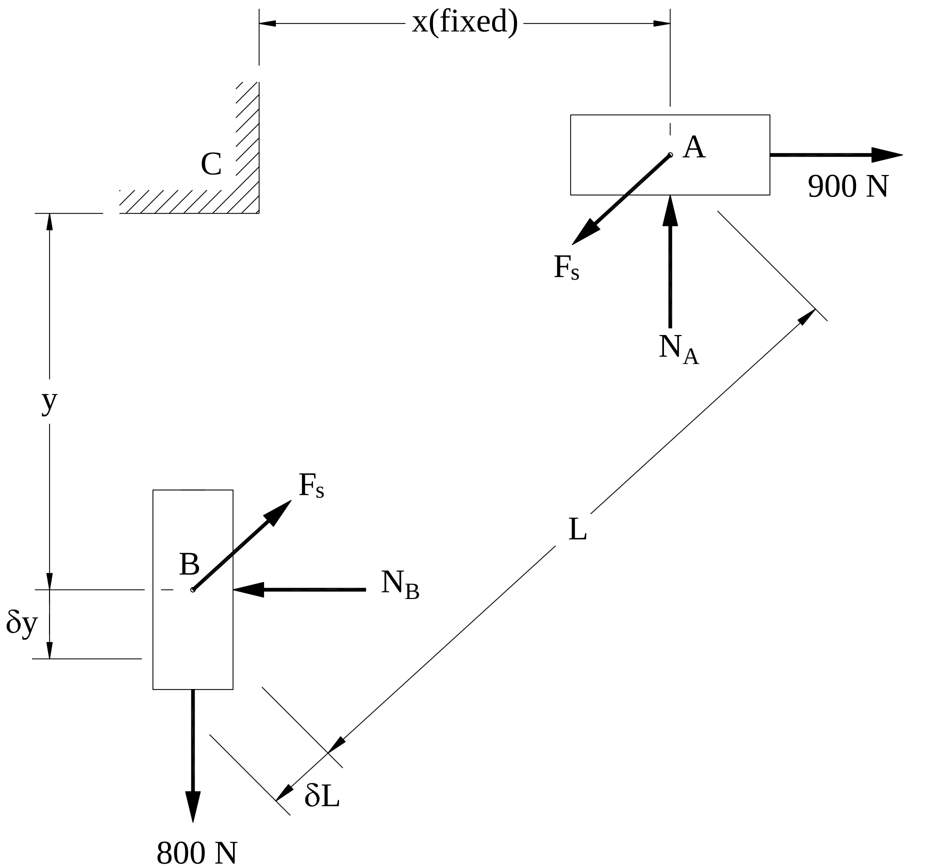
x/L δx

Thus

$$(900 - F_s x/L)\,\delta x = 0$$

Dividing through by δx and re-arranging gives

$$F_s x = 900L \qquad (3)$$



6 Next, hold x fixed and compute the work done when collar B moves an amount δy. Following the same steps as were used for δx leads to

$$F_s y = 800L \qquad (4)$$

The spring force, F_s, is related to L:

$$F_s = k \times \text{extension of the spring}$$

$$= (9000 \text{ N/m}) \times (L - \text{original length})$$

0.2 m

Thus

$$F_s = 9000L - 1800 \qquad (5)$$

We now have four simultaneous nonlinear equations to solve:

$$L^2 = x^2 + y^2 \qquad (2)$$

$$F_s x = 900L \qquad (3)$$

$$F_s y = 800L \qquad (4)$$

$$F_s = 9000L - 1800 \qquad (5)$$

11.1 Virtual Work Example 8, page 5 of 5

7 These equations can be solved directly if a calculator that is able to handle such systems is available.

Alternatively, proceed as follows: square both sides of Eqs. 3 and 4 and add the results to get

$$(F_s x)^2 + (F_s y)^2 = (900L)^2 + (800L)^2.$$

or

$$F_s^2(x^2 + y^2) = L^2(900^2 + 800^2)$$

L^2, by Eq. 2

Solving gives

$$F_s = 1204 \text{ N}$$

Using this result in Eq. 5 gives

$$F_s = 9000L - 1800 \qquad \text{(Eq. 5 repeated)}$$

1204 N

Solving gives

$$L = 0.3338 \text{ m}$$

.

8 Distance x can now be found from Eq. 3:

$$F_s x = 900L \qquad \text{(Eq. 3 repeated)}$$

1204 N 0.3338 m

Solving gives

$$x = 0.250 \text{ m} \qquad \leftarrow\text{Ans.}$$

Distance y can be found from Eq. 4:

$$F_s y = 800L \qquad \text{(Eq. 4 repeated)}$$

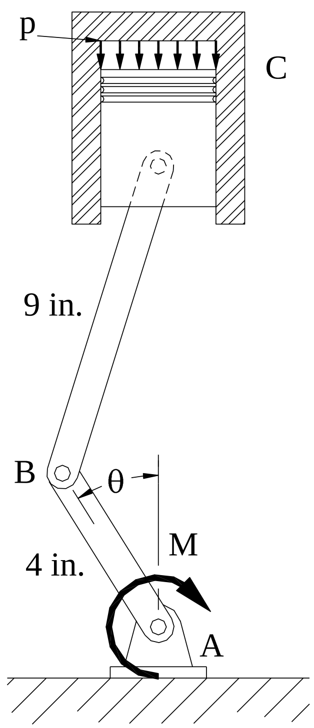
1204 N 0.3338 m

Solving gives

$$y = 0.222 \text{ m} \qquad \leftarrow\text{Ans.}$$

11.1 Virtual Work Example 9, page 1 of 4

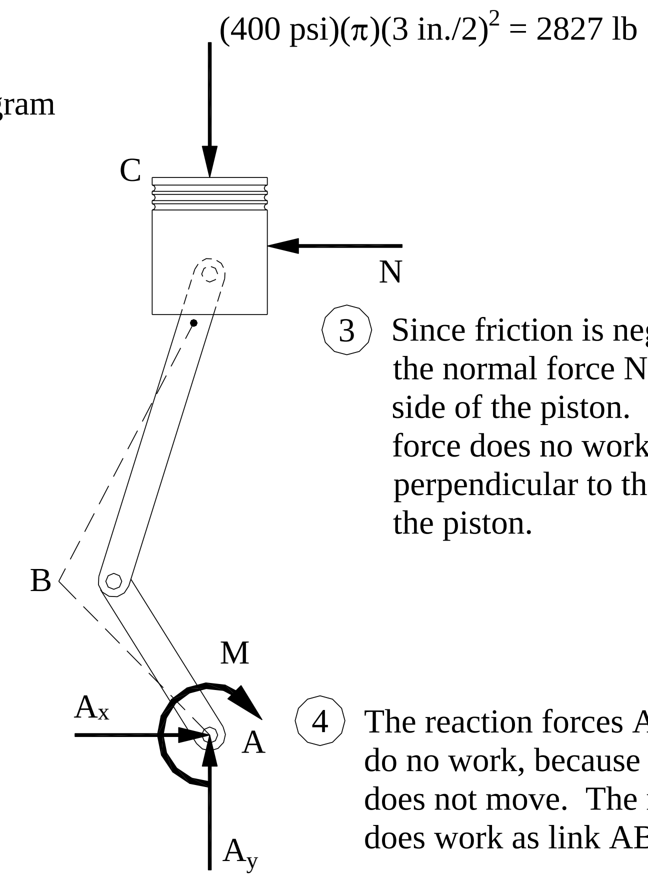
9. Determine the moment M applied to the crankshaft that will keep the piston motionless when a pressure p = 400 psi acts on the top of the piston and $\theta = 25°$. The diameter of the piston is 3 in., and the piston slides with negligible friction in the cylinder.



1 The system can be described by a single coordinate, θ. Consider a free-body diagram and identify the active forces corresponding a small change in θ.

2 The resultant of the pressure is an active force:

Free-body diagram

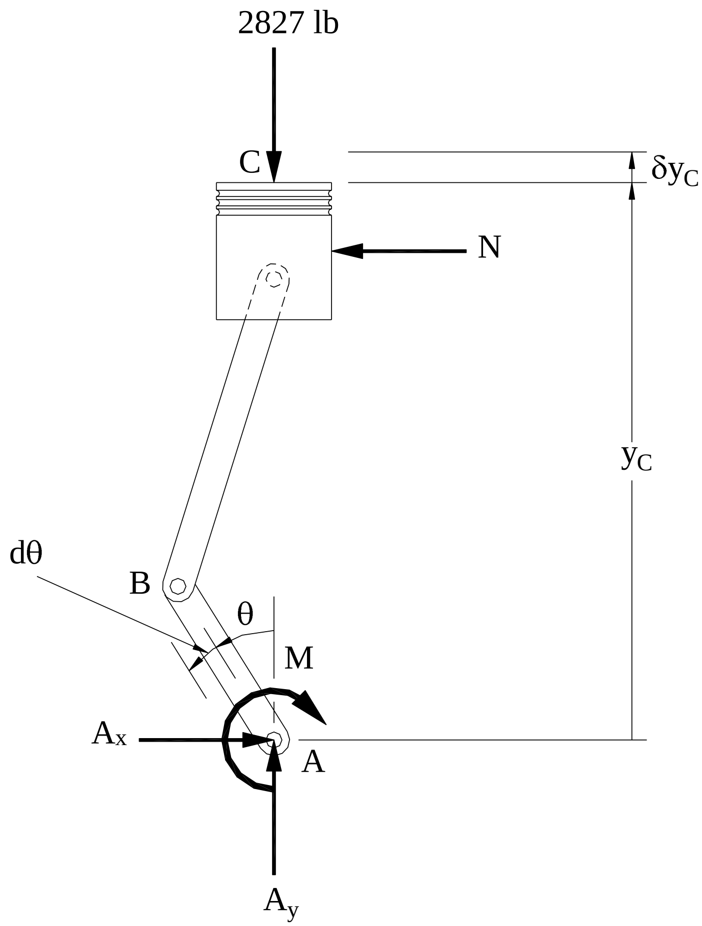


3 Since friction is negligible, only the normal force N acts on the side of the piston. The normal force does no work since it acts perpendicular to the motion of the piston.

4 The reaction forces A_x and A_y do no work, because point A does not move. The moment M does work as link AB rotates.

11.1 Virtual Work Example 9, page 2 of 4

5 Introduce coordinates measured from a fixed reference at point A.



6 Compute the work done when the coordinates are increased a positive infinitesimal amount.

$$\delta U = -(2827 \text{ lb})\, \delta y_C - M\, \delta\theta = 0 \qquad (1)$$

11.1 Virtual Work Example 9, page 3 of 4

7 Relate δy_C to $\delta\theta$:

$$y_C = (4 \text{ in.}) \cos\theta + (9 \text{ in.}) \cos\phi + a$$

$$\delta y_C = -4 \sin\theta\, \delta\theta - 9 \sin\phi\, \delta\phi + \underbrace{\delta a}_{0} \qquad (2)$$

Distance "a" does not change as θ is changed so $\delta a = 0$.

8 Relate θ to ϕ by the law of sines,

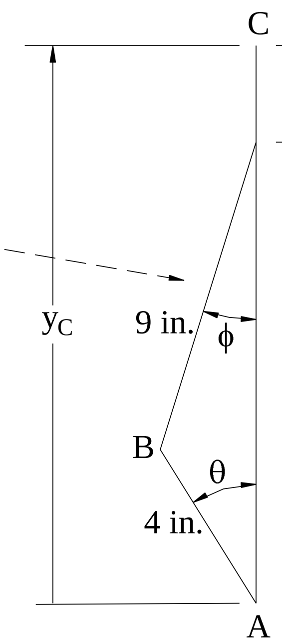
$$\frac{\sin\phi}{4 \text{ in.}} = \frac{\sin\theta}{9 \text{ in.}} \qquad (3)$$

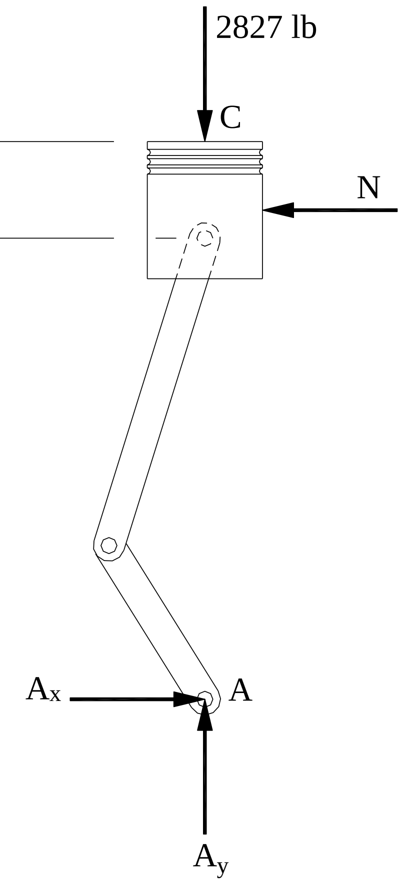
Differentiating gives

$$\frac{\cos\phi\, \delta\phi}{4} = \frac{\cos\theta\, \delta\theta}{9}$$

Thus

$$\delta\phi = \frac{4 \cos\theta\, \delta\theta}{9 \cos\phi} \qquad (4)$$





9 Using Eq. 4 in Eq. 2 gives

$$\delta y_C = -4 \sin\theta\, \delta\theta - 9 \sin\phi\, \delta\phi \qquad \text{(Eq. 2 repeated)}$$

$$\delta\phi = \frac{4 \cos\theta\, \delta\theta}{9 \cos\phi}, \text{ by Eq. 4}$$

$$= (-4 \sin\theta - 4 \tan\phi \cos\theta)\, \delta\theta \qquad (5)$$

Substituting Eq. 5 in the virtual work equation gives

$$-(2827)\, \delta y_C - M\, \delta\theta = 0 \qquad \text{(Eq. 1 repeated)}$$

$$\delta y_C = (-4 \sin\theta - 4 \tan\phi \cos\theta)\, \delta\theta, \text{ by Eq. 5}$$

or

$$[4(2827)(\sin\theta + \tan\phi \cos\theta) - M]\, \delta\theta = 0$$

Dividing through by $\delta\theta$ and solving gives

$$M = 4(2827)(\sin\theta + \tan\phi \cos\theta) \qquad (6)$$

Substituting the given value $\theta = 25°$ into Eq. 3 yields

$$\frac{\sin\phi}{4 \text{ in.}} = \frac{\sin\theta}{9 \text{ in.}} \qquad \text{(Eq. 3 repeated)}$$

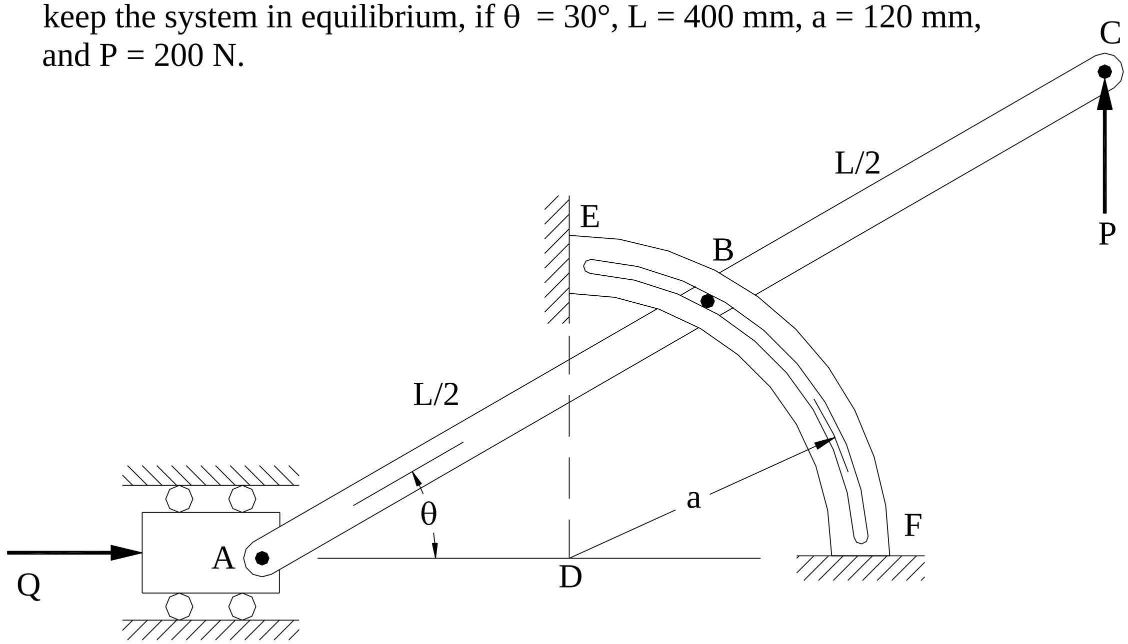
which can be solved to give $\phi = 10.83°$.

10 Using $\theta = 25°$ and $\phi = 10.83°$ in Eq. 6 produces

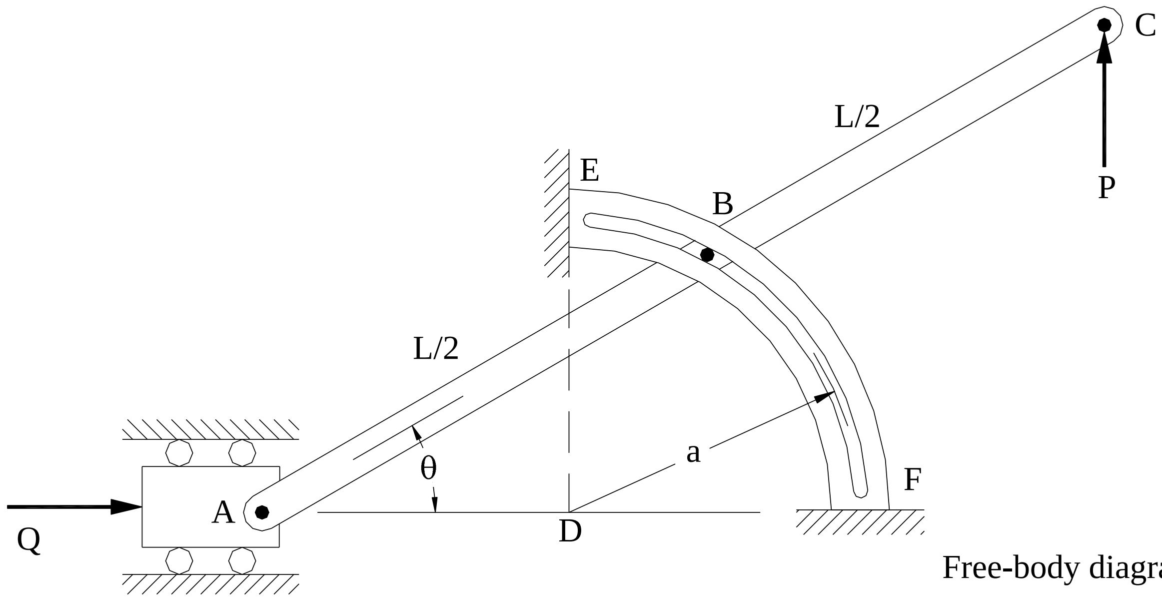
$$M = 6740 \text{ lb}\cdot\text{in} \qquad \leftarrow \text{Ans.}$$

11 Observation: this problem may be more easily solved by using equations of equilibrium. Virtual work is superior to using equations of equilibrium provided that the relation between displacements can be easily obtained. In the present problem, deriving the relation between $\delta\theta$ and δy_C, Eq. 5, was more complicated than simply writing equations of equilibrium.

10. Pin B is rigidly attached to member AC and moves in the smooth quarter-circle slot EF. Determine the value of force Q necessary to keep the system in equilibrium, if θ = 30°, L = 400 mm, a = 120 mm, and P = 200 N.

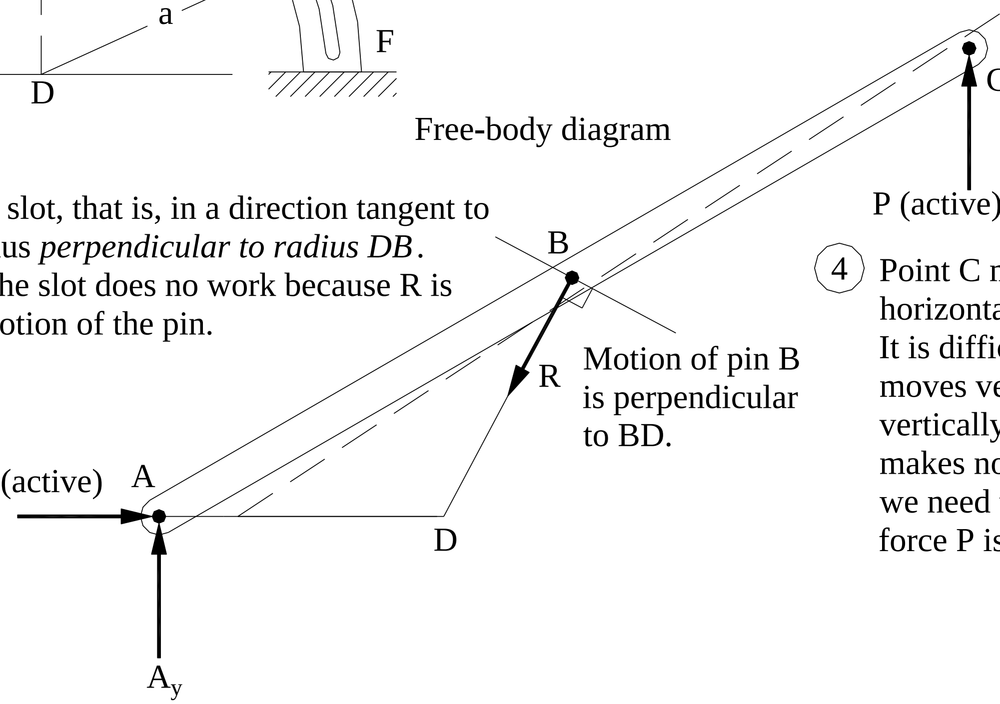


11.1 Virtual Work Example 10, page 2 of 5



1 The system configuration can be defined by the single coordinate, θ. Consider a free-body diagram showing the active forces corresponding to a small increase in θ.

Free-body diagram

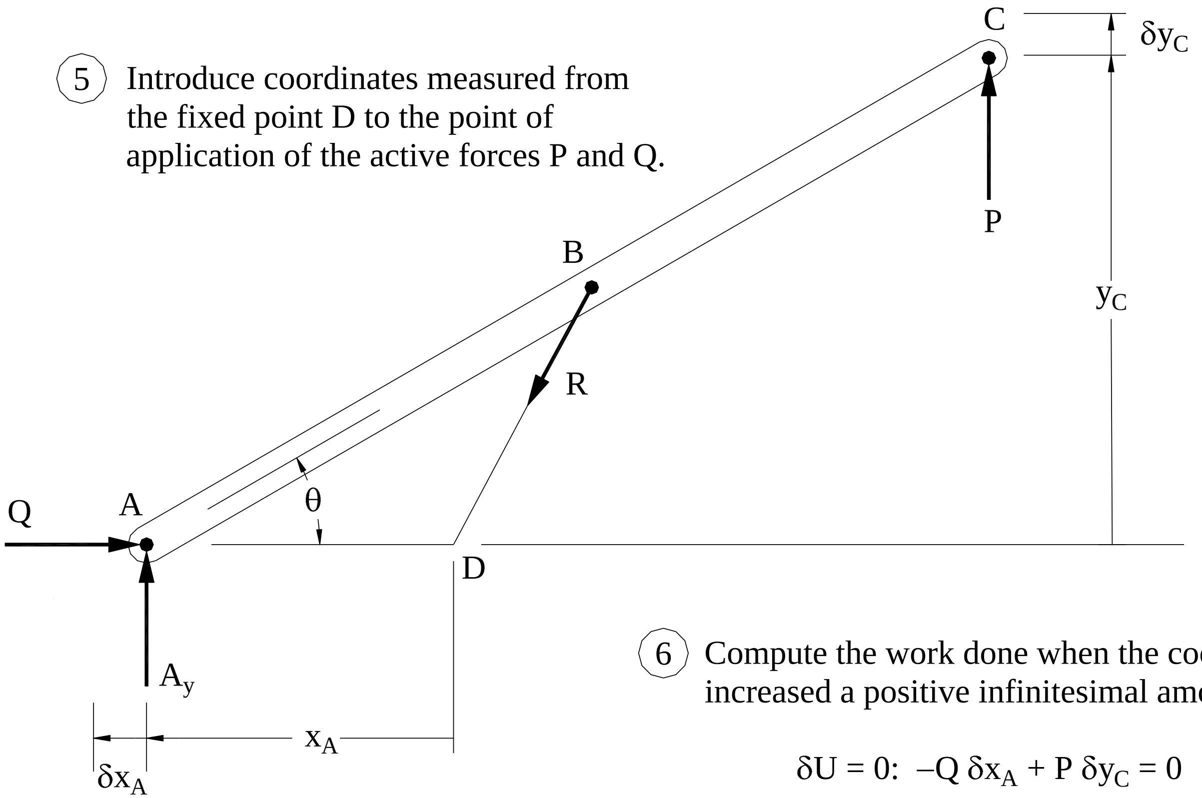


2 Pin B must move in the slot, that is, in a direction tangent to the quarter circle and thus *perpendicular to radius DB*. Thus the force R from the slot does no work because R is *perpendicular* to the motion of the pin.

3 Point A must move *horizontally*. It is difficult to tell if A moves to the right or left, but fortunately it makes no difference. The important thing is to note that force A_y from the rollers does no work so is not an active force.

4 Point C moves both horizontallly and vertically. It is difficult to tell if C moves vertically up or vertically down, but it makes no difference. All we need to note is that force P is an active force.

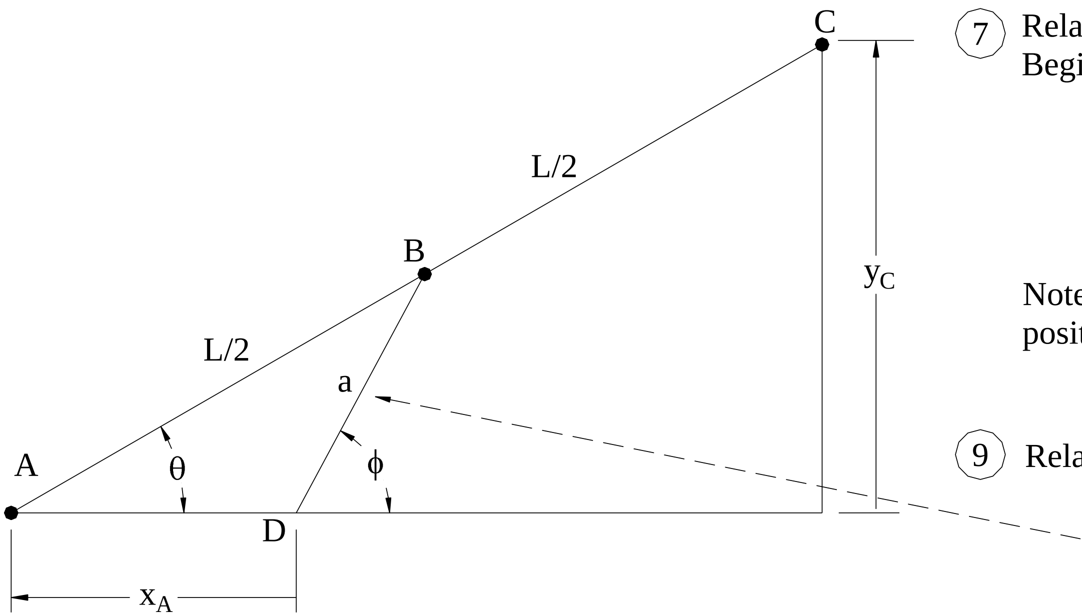
5 Introduce coordinates measured from the fixed point D to the point of application of the active forces P and Q.



6 Compute the work done when the coordinate are increased a positive infinitesimal amount.

$$\delta U = 0: \quad -Q\, \delta x_A + P\, \delta y_C = 0 \tag{1}$$

11.1 Virtual Work Example 10, page 4 of 5



7 Relate the differentials δx_A and δy_C to the angles θ and ϕ. Begin with y_C.

$$y_C = L \sin\theta$$

$$\delta y_C = L \cos\theta \, \delta\theta \qquad (2)$$

Note that this equation shows δy_C is positive if $\delta\theta$ is positive, that is, point C moves up as θ increases.

8 $x_A = (L/2)\cos\theta - a\cos\phi$

$$\delta x_A = -(L/2)\sin\theta\,\delta\theta + a\sin\phi\,\delta\phi \qquad (3)$$

9 Relate θ and ϕ through the law of sines:

$$\frac{\sin(180° - \phi)}{L/2} = \frac{\sin\theta}{a}$$

Because $\sin(180° - \phi) = \sin\phi$, the last equation can be written as

$$a \sin\phi = (L/2)\sin\theta \qquad (4)$$

Differentiating gives

$$a\cos\phi\,\delta\phi = (L/2)\cos\theta\,\delta\theta$$

so

$$\delta\phi = \frac{L\cos\theta\,\delta\theta}{2a\cos\phi} \qquad (5)$$

10 Using Eq. 5 in Eq. 3 gives

$$\delta x_A = -(L/2)\sin\theta\,\delta\theta + a\sin\phi\,\cancel{\delta\phi} \qquad \text{(Eq. 3 repeated)}$$

$$\left(\frac{L\cos\theta\,\delta\theta}{2a\cos\phi}\right)\text{, by Eq. 5}$$

$$= (-\sin\theta + \cos\theta\tan\phi)(L/2)\,\delta\theta \qquad (6)$$

11 The angle ϕ in Eq. 6 can be calculated by substituting the given values $\theta = 30°$, $L = 400$ mm, and $a = 120$ mm into Eq. 4:

$$a \sin\phi = L/2 \sin\theta \qquad \text{(Eq. 4 repeated)}$$

and solving to get $\phi = 56.44°$.

Although it is not necessary for solving the problem, we can now determine whether point A moves to the left or to the right. From Eq. 6 we have

$$\delta x_A = (-\sin\theta + \cos\theta \tan\phi)(L/2)\,\delta\theta \qquad \text{(Eq. 6 repeated)}$$

Substituting $\theta = 30°$ and $\phi = 56.44°$ into this equation gives

$$\delta x_A = (0.8054)(L/2)\,\delta\theta \qquad (7)$$

That is, δx_A is positive when $\delta\theta$ is positive, so x_A increases, that is, point A moves to the left for the particular values of θ, a, and L of our problem.

12 Substituting Eqs. 2 and 7 for δy_C and δx_A into the virtual-work equation, Eq. 1, gives

$$-Q\,\overset{(0.8054)(L/2)\,\delta\theta,\text{ by Eq. 7}}{\cancel{\delta x_A}} + P\,\underset{L\cos\theta\,\delta\theta,\text{ by Eq. 2}}{\cancel{\delta y_C}} = 0 \qquad \text{(Eq. 1 repeated)}$$

so

$$[-Q(0.8054)/2 + P\cos\theta](L\,\delta\theta) = 0$$

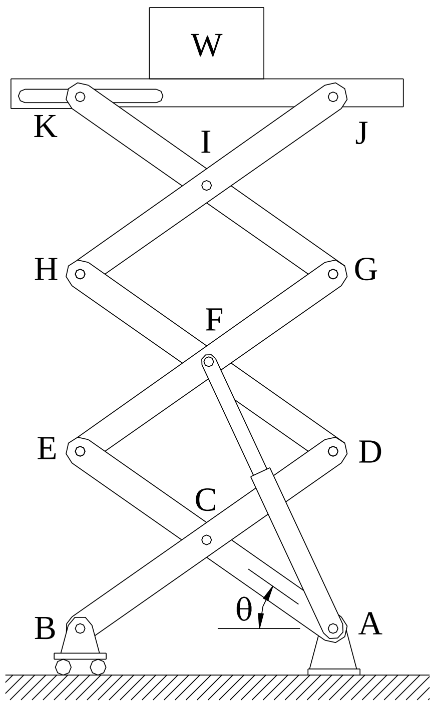
Because $L\,\delta\theta \neq 0$, it follows that

$$-Q(0.8054)/2 + P\cos\theta = 0$$

Substituting $\theta = 30°$ and $P = 200$ N and solving gives

$$Q = 430 \text{ N} \qquad \leftarrow\text{Ans.}$$

11. A scissors lift is used to raise a weight W = 800 lb. Determine the force exerted on pin F by the hydraulic cylinder AF when θ = 35°. Each linkage member is 2-ft long and pin connected at its midpoint and endpoints. The lift consists of two identical linkages and cylinders—the one shown and one directly behind it.

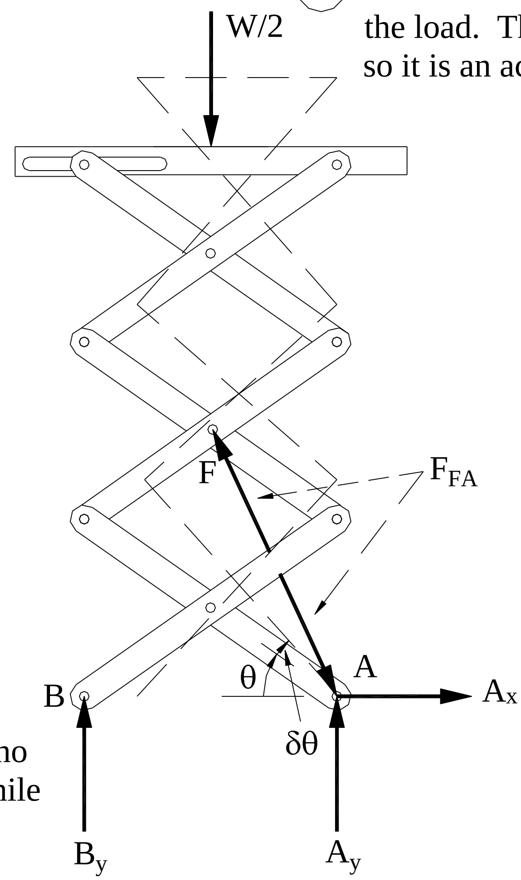


11.1 Virtual Work Example 11, page 2 of 5

1. Consider a free-body diagram and identify the active forces associated with a small change in θ.

Free-body diagram

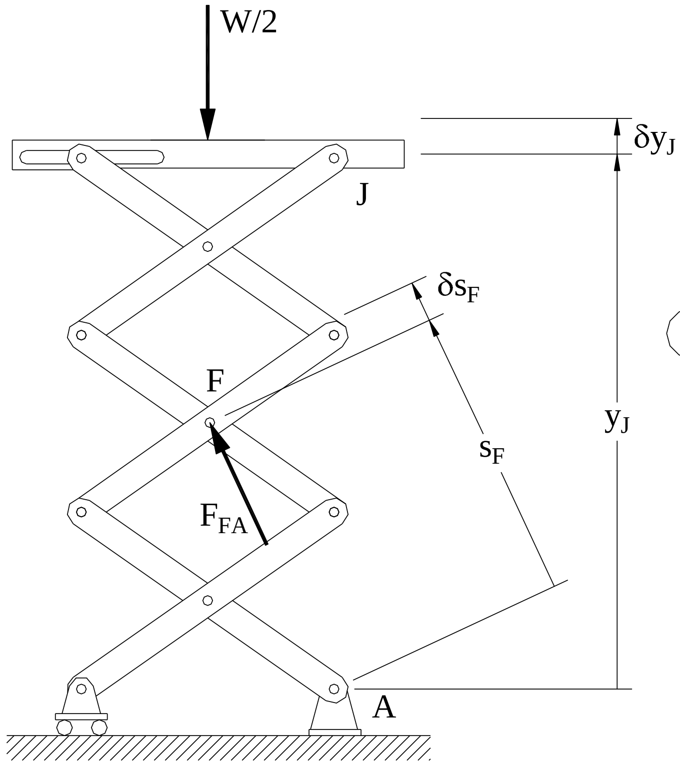
2. Each side of the lift carries half of the load. The W/2 load does work, so it is an active force.



3. The force F_{FA} of the hydraulic cylinder *acting on pin F* does work as pin F moves.

4. The force F_{FA} of the hydraulic cylinder *acting on pin A* does no work because pin A does not move. For the same reason, the reaction forces A_x and A_y from the support do no work.

5. The reaction force at B does no work because it is vertical while the motion of point B is horizontal.

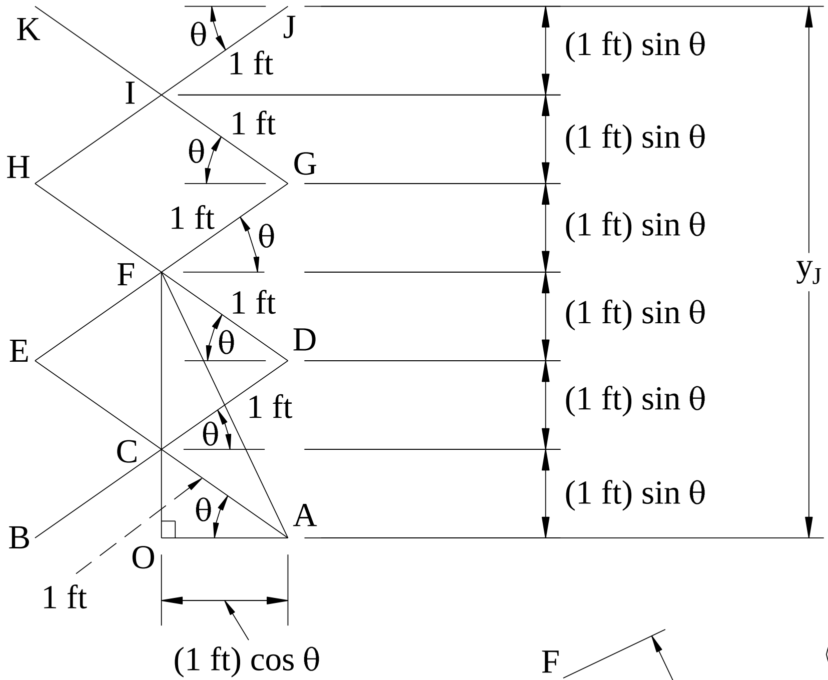


6 Introduce coordinates y_J and s_F measured from the fixed point A to the point of application of the active forces.

Compute the work done when the coordinates are increased a positive infinitesimal amount:

$$\delta U = -(W/2)\, \delta y_J + F_{FA}\, \delta s_F = 0 \tag{1}$$

11.1 Virtual Work Example 11, page 4 of 5

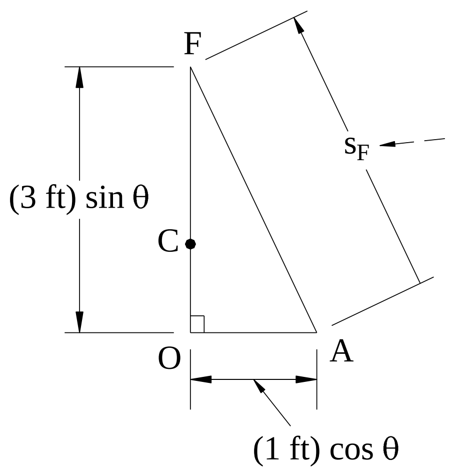


7 Relate the coordinate y_J to the angle θ:

$$y_J = (6\text{ ft}) \sin\theta$$

$$\delta y_J = 6 \cos\theta \, \delta\theta \qquad (2)$$

8 To relate s_F to θ , consider triangle AFCO.



9 Use the Pythagorean Theorem and then differentiate to get δs_F.

$$s_F = \sqrt{(3 \sin\theta)^2 + (1 \cos\theta)^2}$$

$$\delta s_F = \frac{1}{2} \frac{2(3 \sin\theta)(3 \cos\theta)\,\delta\theta + 2(\cos\theta)(-\sin\theta)\,\delta\theta}{\sqrt{(3 \sin\theta)^2 + (1 \cos\theta)^2}}$$

$$= \frac{8 \sin\theta \cos\theta \, \delta\theta}{\sqrt{9 \sin^2\theta + \cos^2\theta}} \qquad (3)$$

10 Substituting for δs_F and δy_J from Eqs. 2 and 3 in the virtual-work equation, Eq. 1, gives

$$-(W/2)\,\overbrace{\delta y_J}^{6\cos\theta\,\delta\theta,\text{ by Eq. 2}} + F_{FA}\,\underbrace{\delta s_F}_{\frac{8\sin\theta\cos\theta\,\delta\theta}{\sqrt{9\sin^2\theta+\cos^2\theta}},\text{ by Eq. 3}} = 0 \qquad \text{(Eq. 1 repeated)}$$

Thus

$$\left[-3W + \frac{8F_{FA}\sin\theta}{\sqrt{9\sin^2\theta+\cos^2\theta}}\right]\cos\delta\theta = 0$$

This implies, since $\cos\theta\,\delta\theta \neq 0$, that

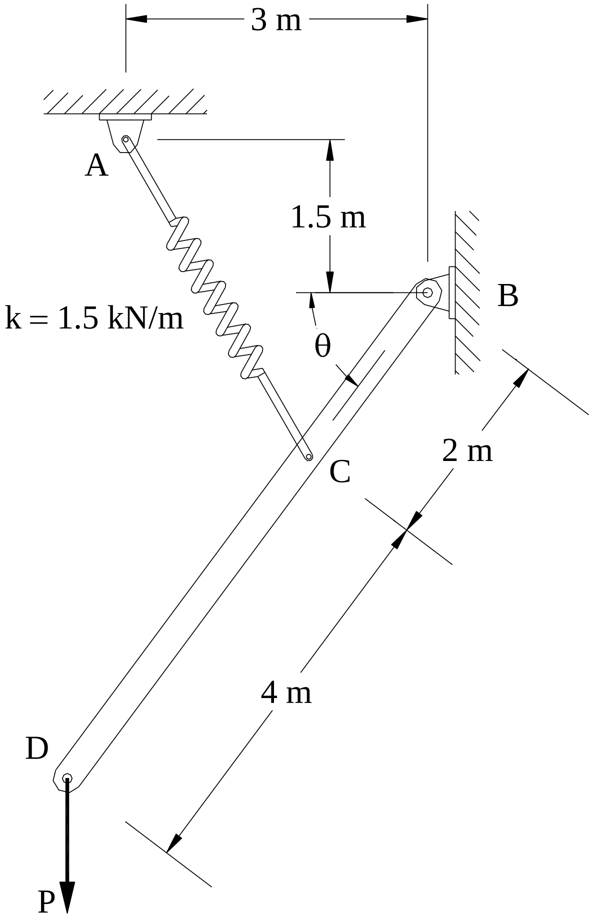
$$-3W + \frac{8F_{FA}\sin\theta}{\sqrt{9\sin^2\theta+\cos^2\theta}} = 0$$

Substituting the given values $\theta = 35°$ and $W = 800$ lb and solving gives

$$F_{FA} = 997\text{ lb} \qquad \leftarrow\text{Ans.}$$

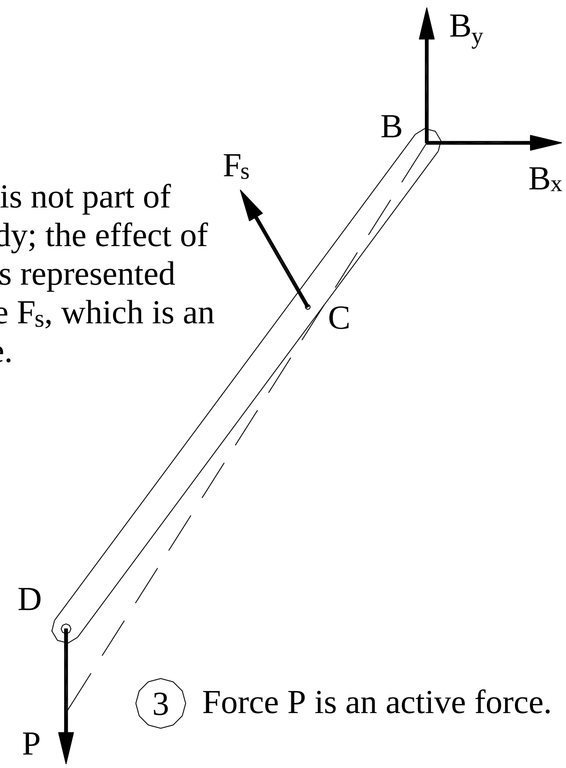
11 Observation: Solving this problem by using the equations of equilibrium would have required drawing several free-body diagrams and writing equations for each diagram. Solving the problem by virtual work is much easier because we don't have to calculate the forces acting between the various links. In general, problems involving connected rigid bodies can be solved more easily by virtual work than by equilibrium equations provided that the relations between displacements can be easily obtained.

12. The unstretched length of the spring is 1 m. Determine the value of θ for equilibrium when force P = 2 kN.



(1) The system can be described by a single coordinate, θ. Consider a free-body diagram and identify the active forces corresponding to a small change in θ.

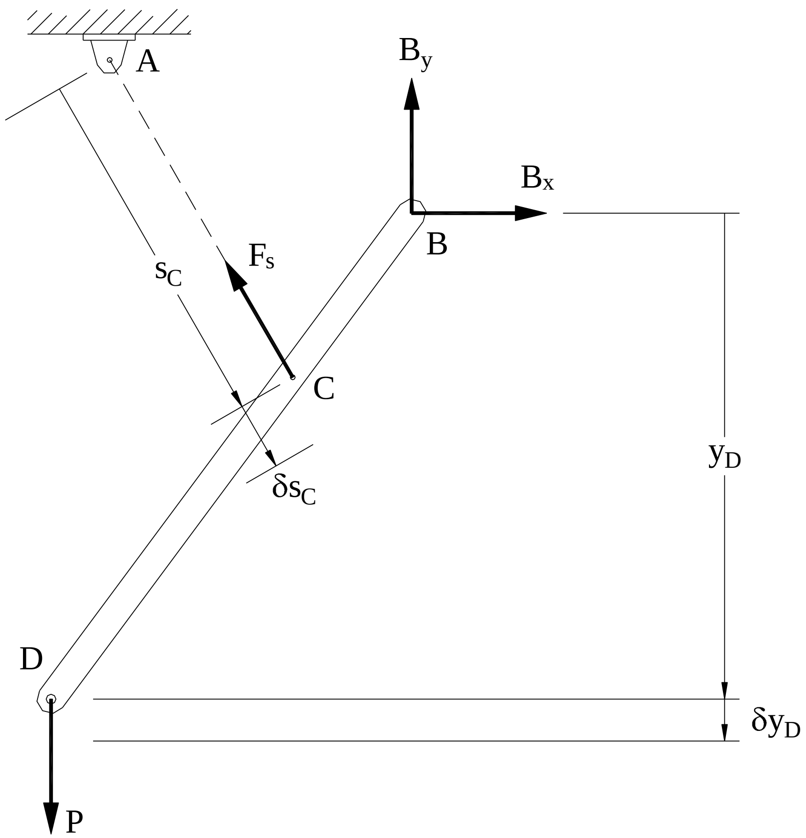
Free-body diagram



(2) The spring is not part of the free-body; the effect of the spring is represented by the force F_s, which is an active force.

(3) Force P is an active force.

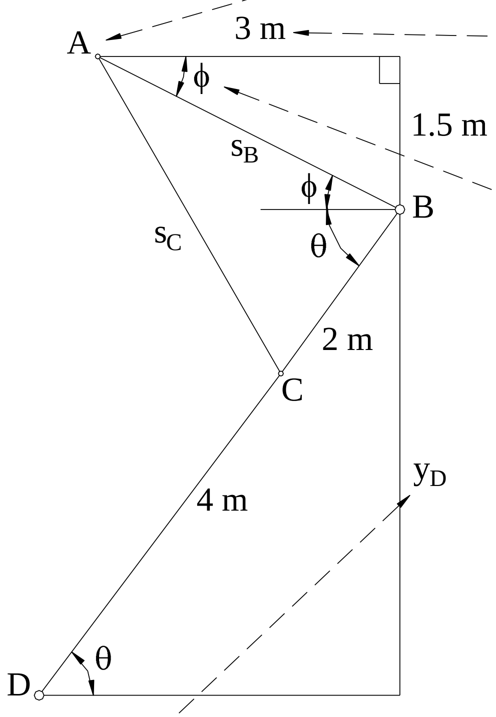
4 Introduce coordinates measured from the fixed points A and B to the point of application of the forces.



5 Compute the work done when the coordinates are increased a positive infinitesimal amount.

$$\delta U = P\,\delta y_D - F_s \delta s_C = 0 \qquad (1)$$

6 Relate the differentials δs_C and δy_D to the angle change $\delta\theta$.



7 $y_D = (2\text{ m} + 4\text{ m}) \sin\theta$

$$\delta y_D = 6 \cos\theta \, \delta\theta \qquad (2)$$

8 Law of cosines applied to triangle ABC:

$$s_C^2 = (2\text{ m})^2 + (s_B)^2 - 2(2\text{ m})(s_B)\cos(\phi + \theta) \qquad (3)$$

Here

$$s_B = \sqrt{(3\text{ m})^2 + (1.5\text{ m})^2} = 3.354\text{ m} \qquad (4)$$

and

$$\phi = \tan^{-1}\frac{1.5\text{ m}}{3\text{ m}} = 26.565° \qquad (5)$$

To avoid having to write equations containing several four and five-digit numbers, introduce intermediate variables a and b:

$$s_C^2 = \underbrace{(2^2 + s_B^2)}_{\equiv\, a} - \underbrace{2(2)(s_B)}_{\equiv\, b}\cos(\phi + \theta) \text{ (Eq. 3 repeated)}$$

Thus

$$s_C^2 = a - b\cos(\phi + \theta) \qquad (6)$$

where

$$a = 2^2 + (\cancel{s_B})^2 \quad \text{3.354 m by Eq. 4}$$

$$= 4 + 3.354^2$$

$$= 15.249\text{ m} \qquad (7)$$

9 The parameter b can also be evaluated, for later use:

$$b = 2(2)(\overset{\text{3.354 m, by Eq. 4}}{\cancel{s_B}})$$

$$= 4(3.354)$$

$$= 13.416 \text{ m} \qquad (8)$$

δs_C can be related to $\delta\theta$ by differentiating Eq. 6:

$$s_C^{\,2} = a - b \cos(\phi + \theta) \qquad \text{(Eq. 6 repeated)}$$

$$2s_C\, \delta s_C = b \sin(\phi + \theta)\, \delta\theta$$

so

$$\delta s_C = \frac{b \sin(\phi + \theta)\, \delta\theta}{2s_C} \qquad (9)$$

Taking the square root of both sides of Eq. 6 gives an equation for s_C.

$$s_C = \sqrt{a - b \cos(\phi + \theta)} \qquad (10)$$

10 The spring force F_s can be expressed in terms of s_C:

$$F_s = k \times (\text{extension of the spring})$$

$$= k \times (\text{stretched length} - \text{unstretched length})$$

$$= k \times (s_C - 1 \text{ m}) \qquad (11)$$

Substituting for δy_D, δs_C, and F_s from Eqs 2, 9, and 11 into the virtual-work equation, Eq. 1, gives

$$P\, \underbrace{\cancel{\delta y_D}}_{6 \cos\theta\, \delta\theta \text{, by Eq. 2}} - \overbrace{\cancel{F_s}}^{k(s_C - 1)\text{, by Eq. 11}}\, \underbrace{\cancel{\delta s_C}}_{\frac{b \sin(\phi + \theta)\, \delta\theta}{2s_C}\text{, by Eq. 9}} = 0 \qquad \text{(Eq. 1 repeated)}$$

or

$$\left[(6P) \cos\theta - k(s_C - 1) \frac{b \sin(\phi + \theta)}{2s_C}\right] \delta\theta = 0$$

Since $\delta\theta \neq 0$, the expression in brackets must equal to zero.

$$(6P) \cos\theta - k(s_C - 1) \frac{b \sin(\phi + \theta)}{2s_C} = 0 \qquad (12)$$

11 Eq. 12 contains the distance s_C, which can be calculated by using Eq. 10:

$$(6P)\cos\theta - k(s_C - 1)\frac{b\sin(\phi+\theta)}{2s_C} = 0 \qquad \text{(Eq. 12 repeated)}$$

$$\sqrt{a - b\cos(\phi+\theta)}\text{, by Eq. 10}$$

or

$$(6P)\cos\theta - k(\sqrt{a - b\cos(\phi+\theta)} - 1)\frac{b\sin(\phi+\theta)}{2\sqrt{a - b\cos(\phi+\theta)}} = 0$$

Substituting in the latter equation the values

$P = 2$ kN (Given)

$k = 1.5$ kN/m (Given)

$a = 15.249$ m (Eq. 7 repeated)

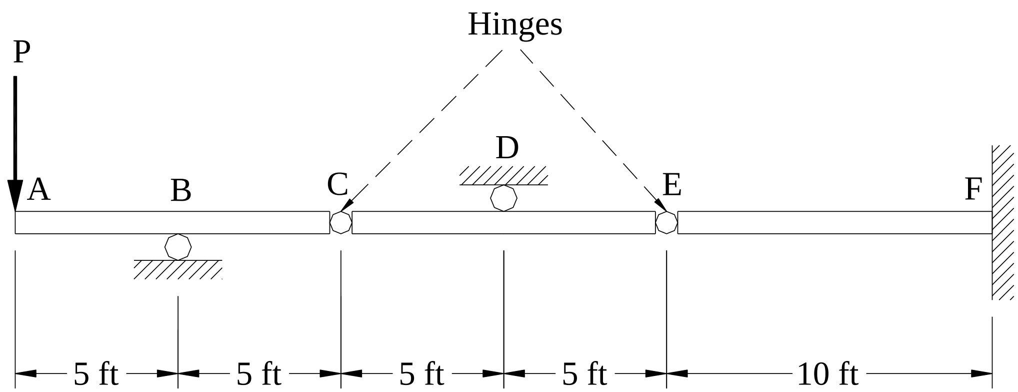
$b = 13.416$ m (Eq. 8 repeated)

$\phi = 26.565°$ (Eq. 5 repeated)

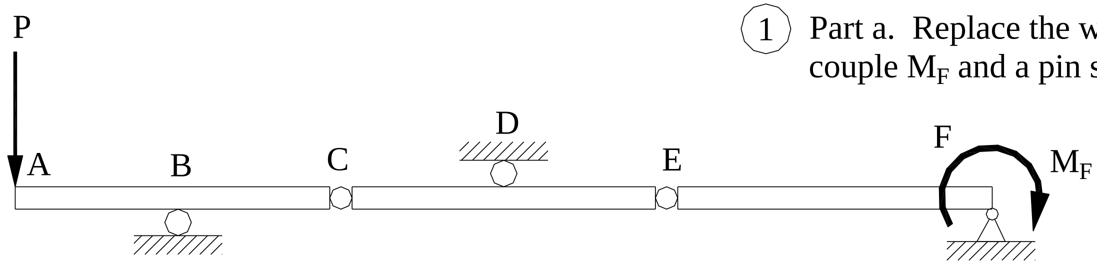
and solving numerically gives

$\theta = 53.4$ ←Ans.

13. a) Determine the moment reaction at the wall F.
   b) Determine the force reaction at the roller D.
   In both cases P = 60 lb.

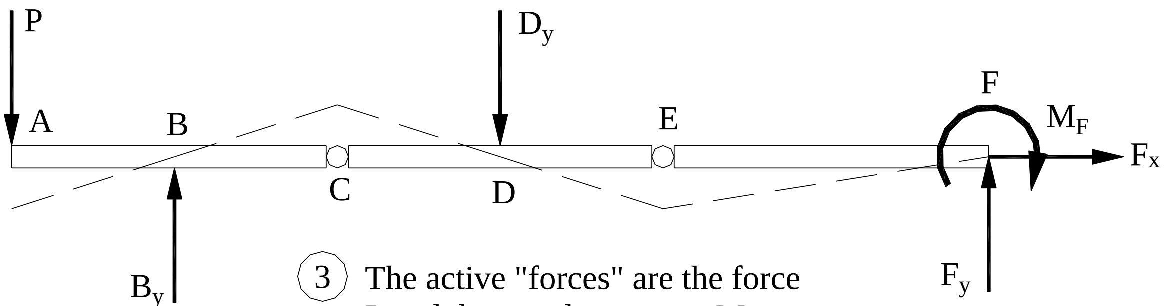


(1) Part a. Replace the wall at F by a moment couple M_F and a pin support.



2 Consider a free-body diagram and identify the active forces associated with a small rotation of the segments of the beam.

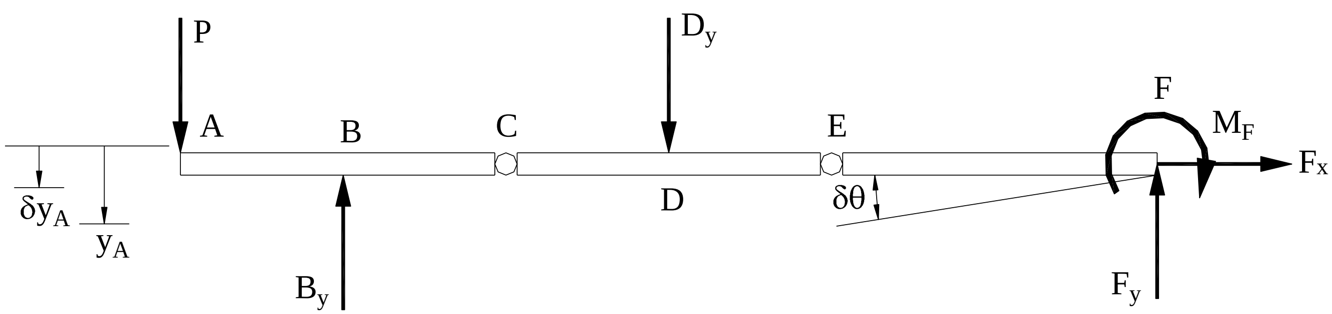
Free-body diagram



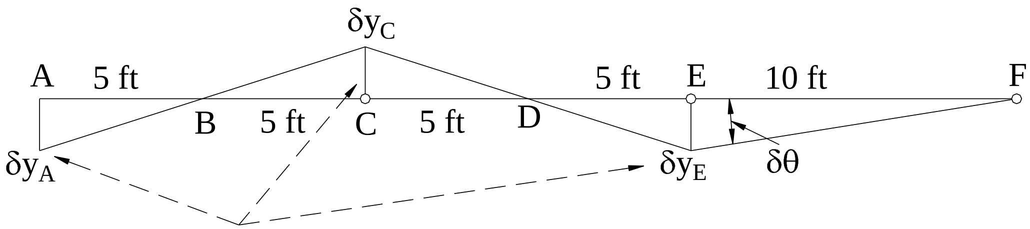
3 The active "forces" are the force P and the couple moment M_F.

4 Introduce coordinates y_A and θ for calculating the work.

$$\delta U = 0: \; P\,\delta y_A - M_F\,\delta\theta = 0 \qquad (1)$$



5 Relate the differentials δy_A and $\delta\theta$.



6 By similar triangles,

$$\delta y_E = \delta y_C \text{ and } \delta y_C = \delta y_A$$

so

$$\delta y_E = \delta y_A$$

7 For small angles, δy_E is given by

$$\delta y_E = (10 \text{ ft})\ \delta\theta$$

But this becomes, after using the relation $\delta y_E = \delta y_A$,

$$\delta y_A = 10\ \delta\theta \qquad (2)$$

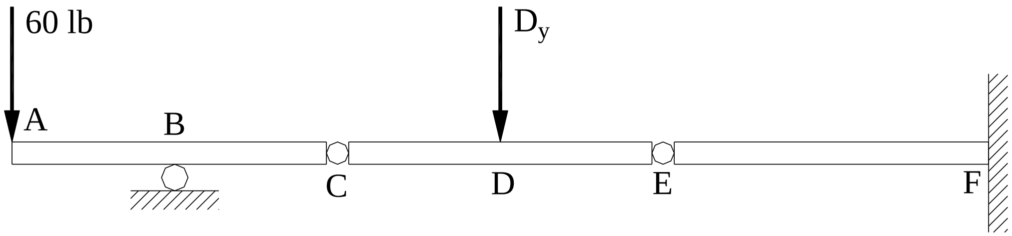
Substitute this result in the virtual-work equation to get

$$P\ \underbrace{\delta y_A}_{10\ \delta\theta} - M_F\ \delta\theta = 0 \qquad \text{(Eq. 1 repeated)}$$

Dividing through by $\delta\theta$, substituting the known value $P = 60$ lb, and then solving for M_F gives,

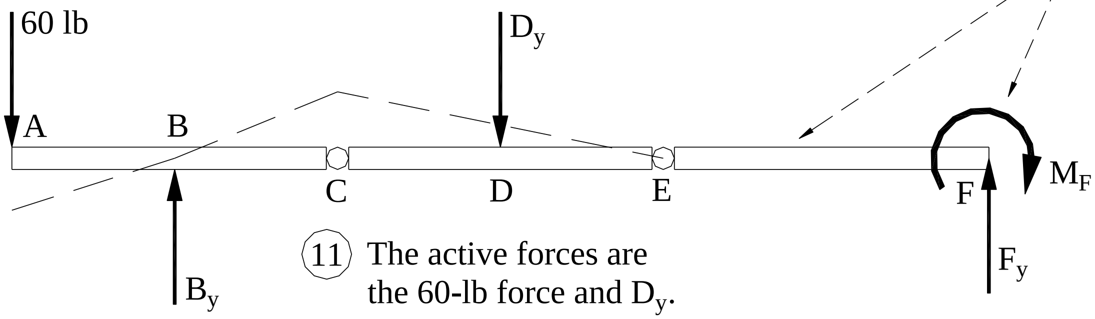
$$M_F = 600 \text{ lb}\cdot\text{ft} \qquad \leftarrow\text{Ans.}$$

8 Part b. Replace the roller at D by a vertical force, D_y.



9 Draw a free-body diagram and show a small rotation of segments AC and CE.

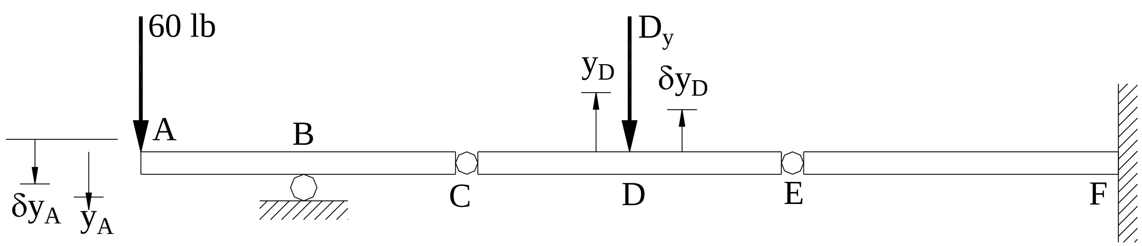
10 Segment EF of the beam does not move because the wall support prevents both rotation and vertical displacement. Thus M_F and F_y do no work.



11 The active forces are the 60-lb force and D_y.

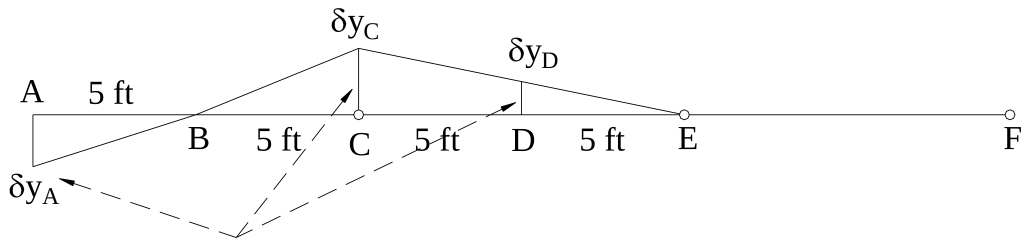
12 Introduce coordinates y_A and y_D for calculating the work.

$$\delta U = 0\text{:}\quad (60\text{ lb})\,\delta y_A - D_y\,\delta y_D = 0 \qquad (3)$$



11.1 Virtual Work Example 13, page 5 of 6

13 Relate the differentials δy_A and δy_D.



14 By similar triangles,

$$\delta y_A = \delta y_C$$

and

$$\frac{\delta y_C}{5 + 5} = \frac{\delta y_D}{5}$$

Eliminating δy_C gives

$$\delta y_D = \frac{\delta y_A}{2}$$

Use this equation to replace δy_D in the virtual-work equation:

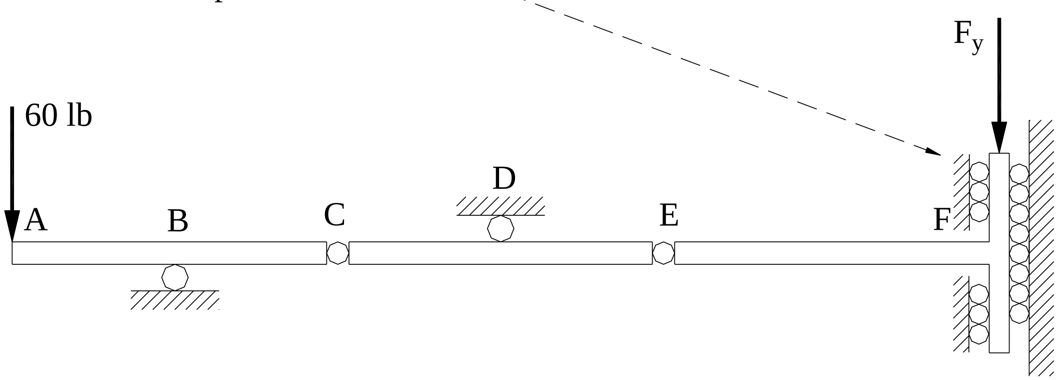
$$(60)\,\delta y_A - D_y \underset{\frac{\delta y_A}{2}}{\cancel{\delta y_D}} = 0 \qquad \text{(Eq. 3 repeated)}$$

Dividing through by δy_A and solving gives

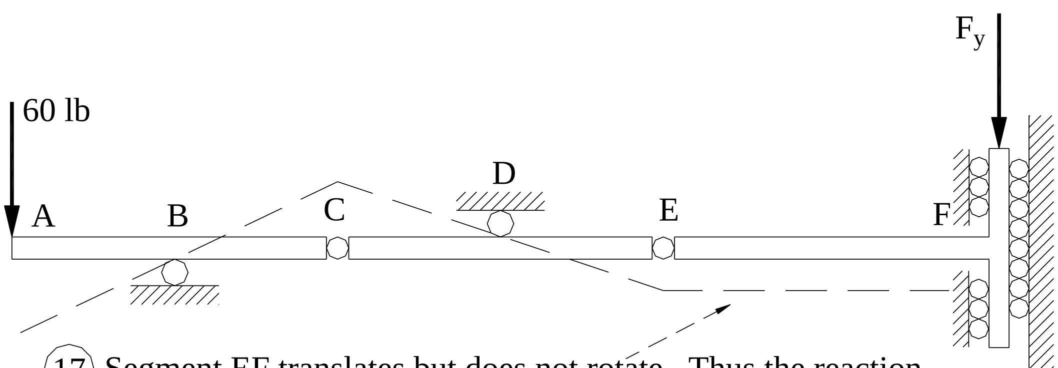
$$D_y = 120 \text{ lb} \qquad \leftarrow \text{Ans.}$$

11.1 Virtual Work Example 13, page 6 of 6

15 Comment: Let's extend the discussion. If we were asked to calculate the vertical reaction force at the wall, we would replace the wall by a support that prevents rotation but permits vertical displacement.



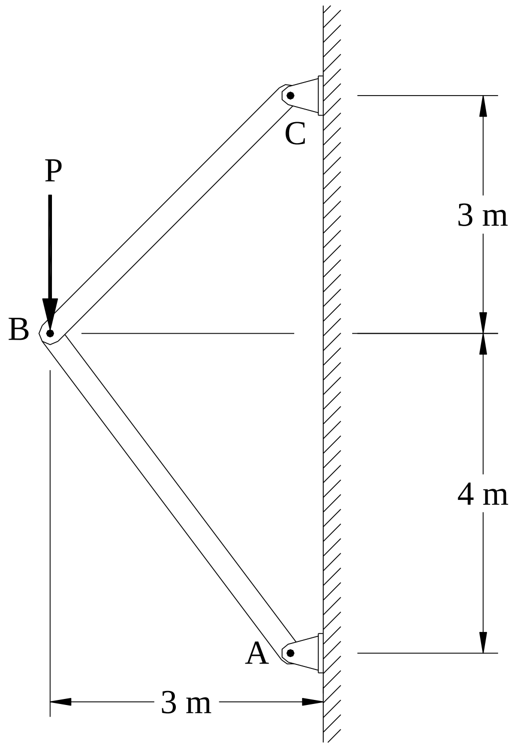
16 Corresponding displacements



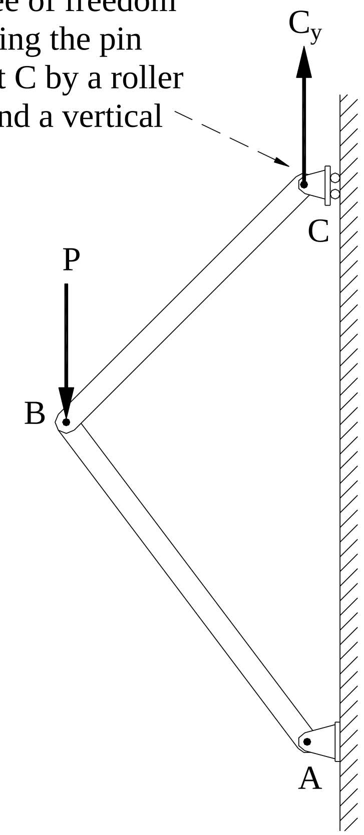
17 Segment EF translates but does not rotate. Thus the reaction moment at the support does no work. The reaction force, F_y, however, does work and is thus an active force.

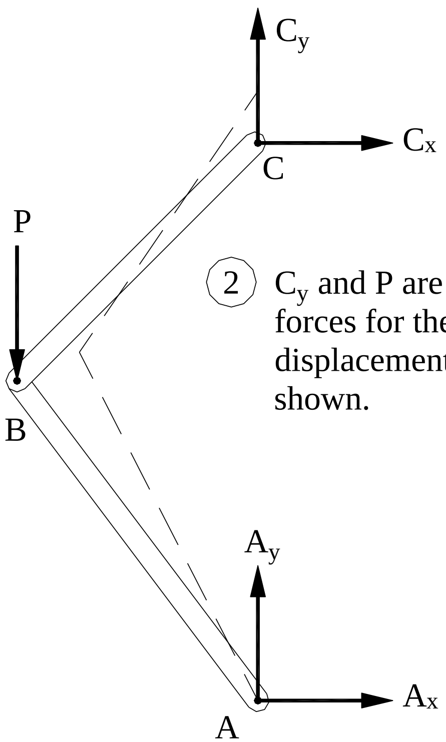
18 Observation: The method applied in this beam example can be generalized. Virtual work can be used to calculate a force of constraint (a reaction) by considering displacements *which violate the constraints* and then accounting for the work done by the force of constraint. This procedure is equivalent to converting a rigid structure into a mechanism, as was done at the beginning of the present example.

14. Determine the vertical reaction at support C, if P = 2 kN.

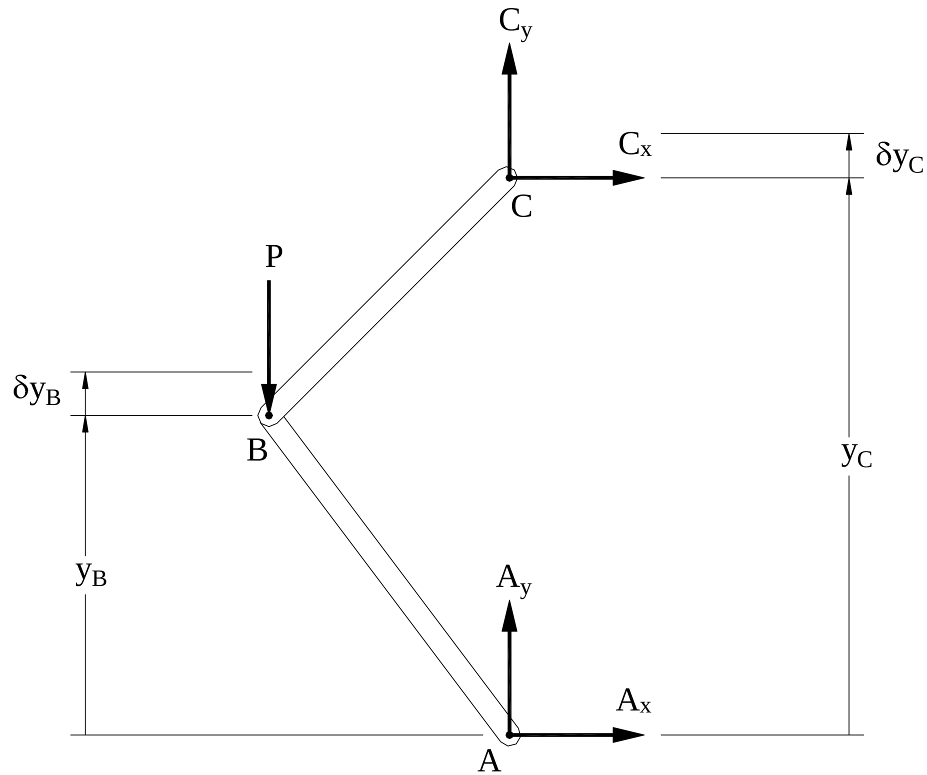


1 Convert the structure into a mechanism with one degree of freedom by replacing the pin support at C by a roller support and a vertical force, C_y.





(2) C_y and P are active forces for the displacements shown.

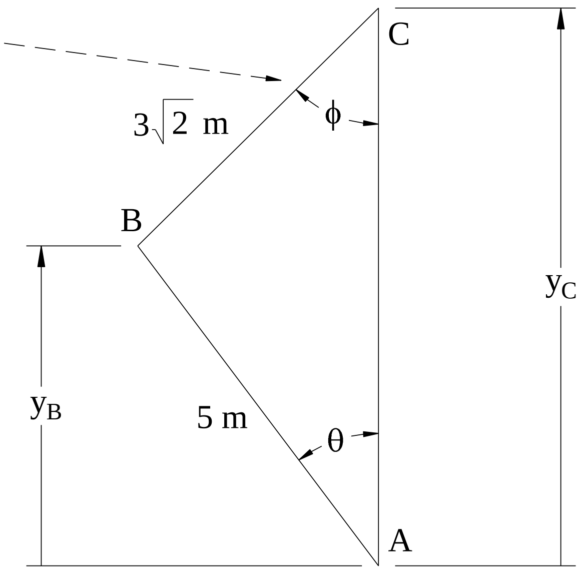


(3) Define coordinates y_B and y_C locating the point of application of the active forces, and compute the work.

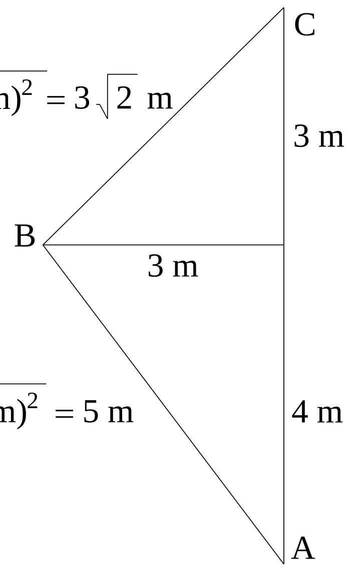
$$\delta U = 0: \ -P\,\delta y_B + C_y\,\delta y_C = 0 \qquad (1)$$

11.1 Virtual Work Example 14, page 3 of 5

4 Relate the differentials δy_B and δy_C through the angles θ and ϕ. Begin by computing the lengths of bars BC and BA (Note that these lengths do not change, as the angles θ and ϕ change).



5 $BC = \sqrt{(3\text{ m})^2 + (3\text{ m})^2} = 3\sqrt{2}\text{ m}$



6 $AB = \sqrt{(3\text{ m})^2 + (4\text{ m})^2} = 5\text{ m}$

7 From the above figure,

$$y_B = (5\text{ m})\cos\theta$$

$$\delta y_B = -5\sin\theta\,\delta\theta \qquad (2)$$

$$y_C = (5\text{ m})\cos\theta + (3\sqrt{2}\text{ m})\cos\phi$$

$$\delta y_C = -5\sin\theta\,\delta\theta - (3\sqrt{2})\ \sin\phi\,\delta\phi \qquad (3)$$

Use the law of sines to relate ϕ and θ

$$\frac{\sin\phi}{5\text{ m}} = \frac{\sin\theta}{3\sqrt{2}\text{ m}} \qquad (4)$$

8 Differentiating Eq. 4 gives

$$\frac{\cos\phi}{5}\,\delta\phi = \frac{\cos\theta}{3\sqrt{2}}\,\delta\theta$$

Thus

$$\delta\phi = \frac{5\cos\theta\ \delta\theta}{3\sqrt{2}\ \cos\phi} \qquad (5)$$

The equation relating $\delta\phi$ and $\delta\theta$, Eq. 5, can be used in Eq. 3 to express δy_C in terms of $\delta\theta$ alone:

$$\delta y_C = -5\sin\theta\,\delta\theta - 3\sqrt{2}\,\sin\phi\,\underbrace{\cancel{\delta\phi}}_{\frac{5\cos\theta\,\delta\theta}{3\sqrt{2}\,\cos\phi}} \qquad \text{(Eq. 3 repeated)}$$

$$= (-5\sin\theta - 5\tan\phi\cos\theta)\,\delta\theta \qquad (6)$$

Substituting Eqs. 2 and 6 for δy_B and δy_C into the virtual work equation gives

$$-P\,\overbrace{\cancel{\delta y_B}}^{-5\sin\theta\,\delta\theta,\ \text{by Eq. 2}} + C_y\,\underbrace{\cancel{\delta y_C}}_{(-5\sin\theta - 5\tan\phi\cos\theta)\,\delta\theta,\ \text{by Eq. 6}} = 0 \qquad \text{(Eq. 1 repeated)}$$

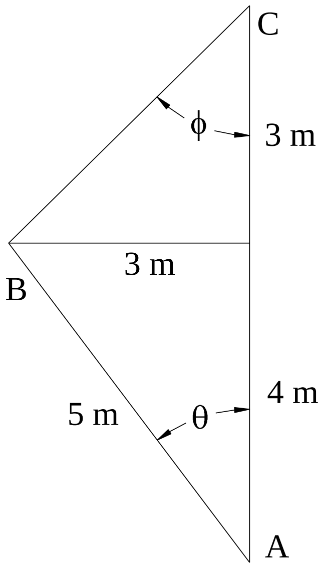
9 Thus

$$[P\sin\theta - C_y(\sin\theta + \tan\phi\cos\theta)](5\,\delta\theta) = 0$$

Dividing by $5\,\delta\theta$ gives

$$P\sin\theta - C_y(\sin\theta + \tan\phi\cos\theta) = 0 \qquad (7)$$

11.1 Virtual Work Example 14, page 5 of 5



$$\tan\phi = \frac{3}{3} = 1$$

$$\sin\theta = \frac{3}{5}$$

$$\cos\theta = \frac{4}{5}$$

10 Evaluating the functions of ϕ and θ in Eq. 7, substituting the given value P = 2 kN, and then solving gives

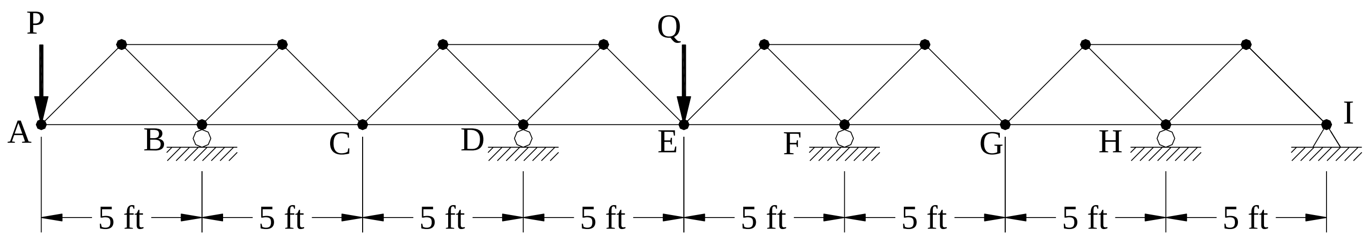
$$P \cancelto{\frac{3}{5}}{\sin\theta} - C_y \left(\cancelto{\frac{3}{5}}{\sin\theta} + \cancelto{1}{\tan\phi}\, \cancelto{\frac{4}{5}}{\cos\theta}\right) = 0 \qquad \text{(Eq. 7 repeated)}$$

$$C_y = 0.857 \text{ kN} \qquad \leftarrow \text{Ans.}$$

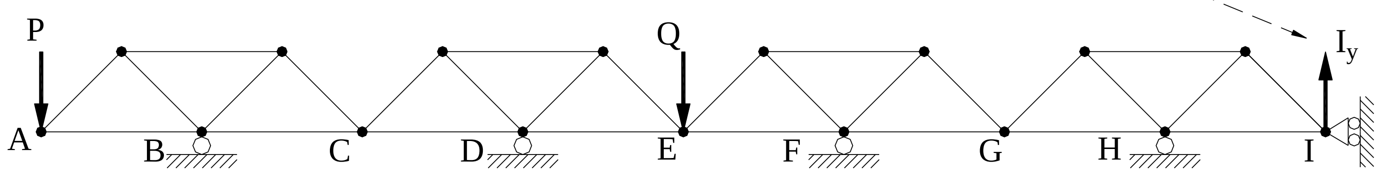
11 Observation: This example demonstrates that virtual work can be used to calculate the reaction forces from the supports acting on a structure. The example also demonstrates that just because virtual work *can* be used doesn't necessarily mean that it *should* be used—the reaction at support C could have been found much more easily by employing equilibrium equations. The usefulness of virtual work depends on how easy it is to express relations between coordinates.

11.1 Virtual Work Example 15, page 1 of 3

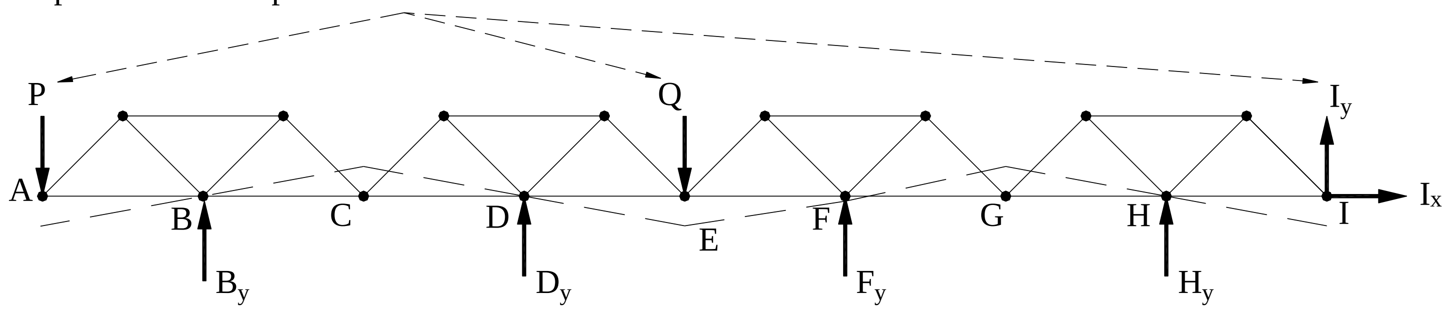
15. Determine the vertical reaction at support I of the truss, if P = 10 kip = Q.

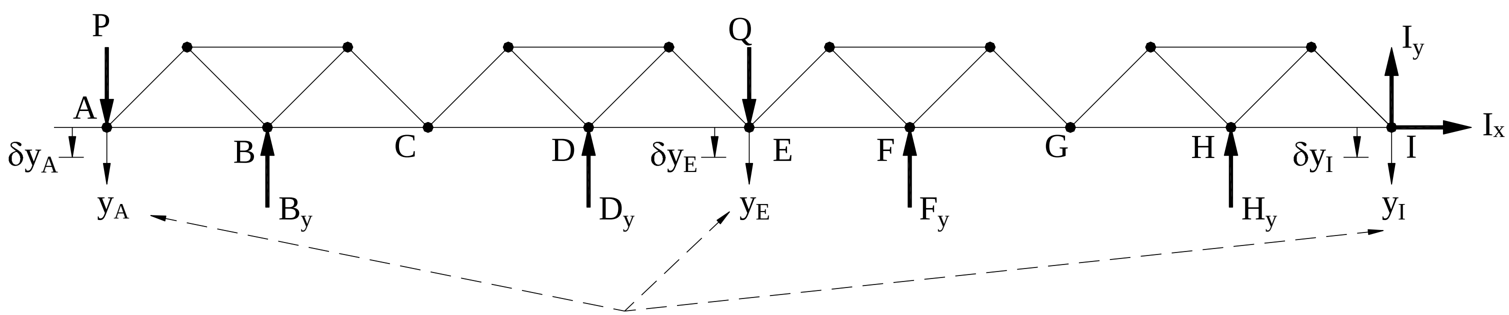


(1) Convert the structure to a mechanism with one degree of freedom by replacing the pin support at I by a vertical force I_y and a roller.



2 Identify the active forces corresponding to a set of displacements compatible with the constraints.



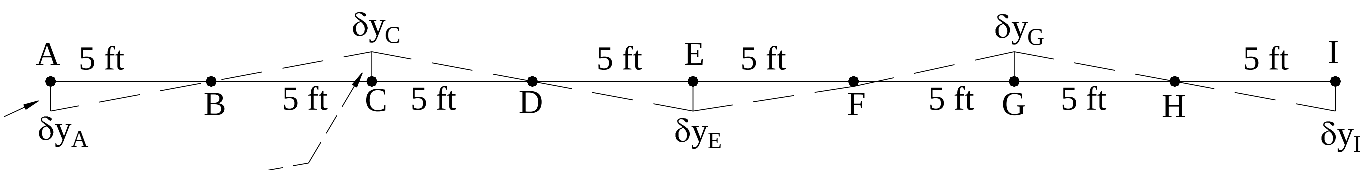


3 Introduce coordinates measured from fixed points to the points of application of the applied forces.

Calculate the work done.

$$\delta U = 0: \; P\,\delta y_A + Q\,\delta y_E - I_y\,\delta y_I = 0 \qquad (1)$$

4 Relate δy_A, δy_E, and δy_I by geometry (similar triangles).



$$\delta y_A = \delta y_C \qquad (2)$$

Similarly,

$$\delta y_C = \delta y_E,\ \delta y_E = \delta y_G, \text{ and } \delta y_G = \delta y_I$$

These equations imply

$$\delta y_A = \delta y_I \text{ and } \delta y_E = \delta y_I$$

Substituting the latter pair of equations into the virtual-work equation, Eq. 1, gives

$$P\ \cancelto{\delta y_I}{\delta y_A} + Q\ \cancelto{\delta y_I}{\delta y_E} - I\,\delta y_I = 0 \qquad \text{(Eq. 1 repeated)}$$

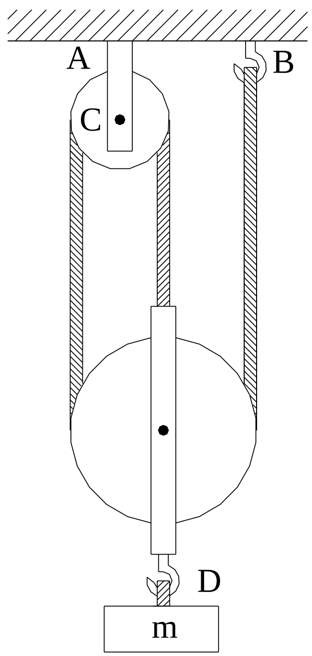
or

$$(P + Q - I_y)\,\delta y_I = 0$$

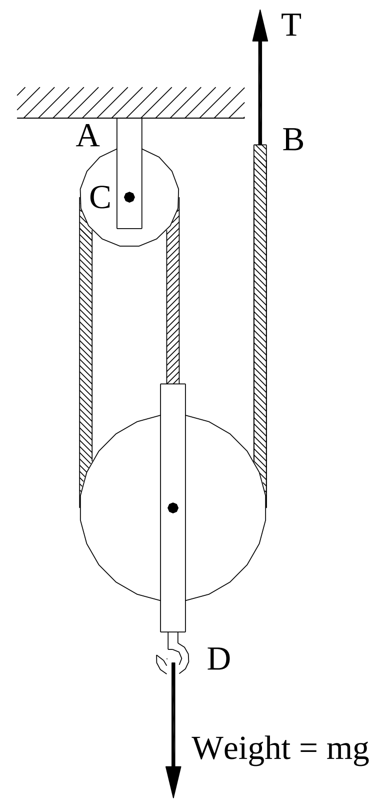
Dividing through by δy_I, substituting the given values P = 10 kip = Q, and solving gives

$$I_y = 20 \text{ kip} \qquad \leftarrow\text{Ans.}$$

16. Determine the tension in the cord. The pulleys are frictionless and m = 90 kg.



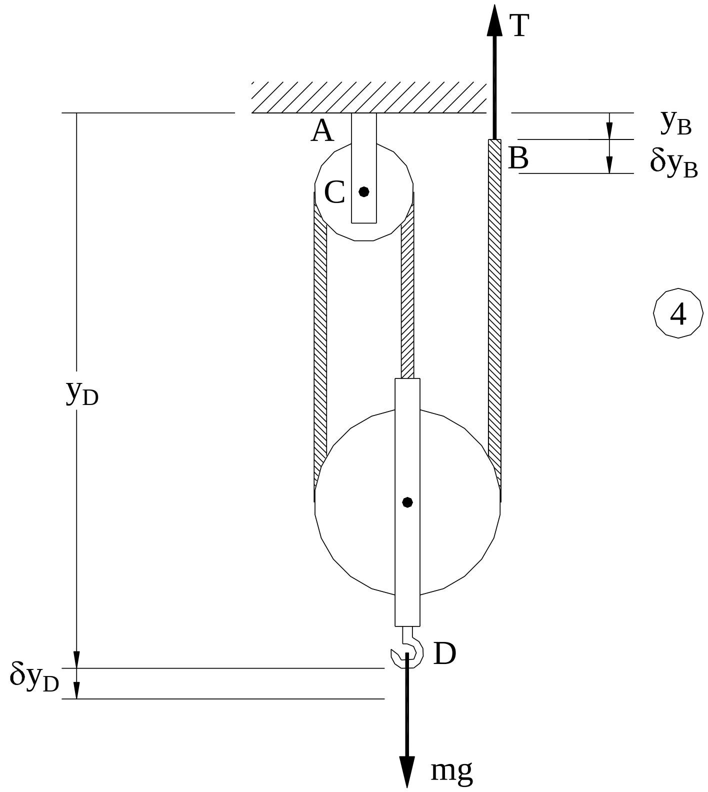
1 Convert the pulley-cord system into a mechanism with one degree of freedom by replacing the support B by a tensile force T acting on the end of the cord.



11.1 Virtual Work Example 16, page 2 of 4

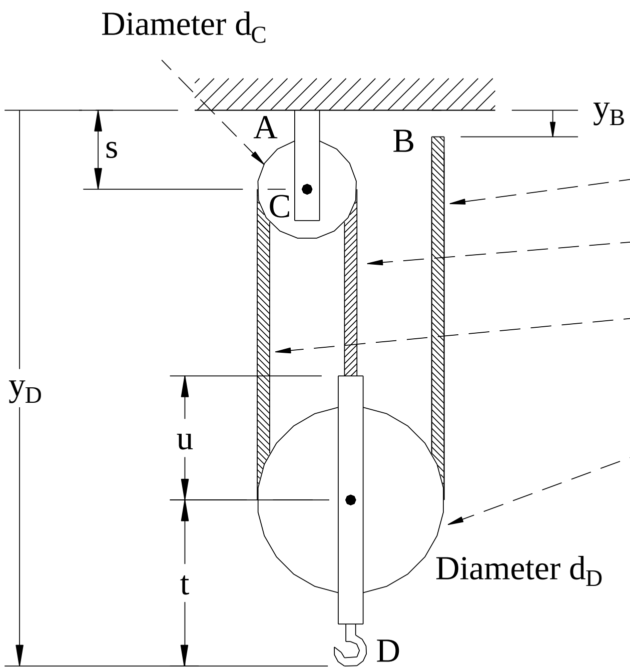
2 The tension T and the weight do work (are active forces) if end B of the cord moves up a small amount.

3 Introduce the coordinates y_B and y_D.



4 Calculate the work done:

$$\delta U = 0: \; -T\,\delta y_B + mg\,\delta y_D = 0 \qquad (1)$$



5 Relate δy_B to δy_D by first expressing the length, say L, of the cord in terms of y_B and y_D:

$$
\begin{aligned}
L = {} & [(y_D - t) - y_B] \\
& + [(y_D - t) - u - s] \\
& + [(y_D - t) - s] \\
& \left.\begin{aligned} & + \pi d_D/2 \\ & + \pi d_C/2 \end{aligned}\right\} \text{Half circumference of pulleys}
\end{aligned}
$$

Thus

$$L = 3y_D - 3t - y_B - u - 2s + \pi d_D/2 + \pi d_C/2$$

Now differentiate, taking into account that because L, t, u, s, d_D, and d_C do not vary as y_D and y_B vary, we have $\delta L = 0 = \delta t = \delta u = \delta s = \delta d_D = \delta d_C$; the result of the differentiation is, then,

$$0 = 3\delta y_D - \delta y_B$$

Thus

$$\delta y_B = 3\delta y_D \qquad (2)$$

6 Substituting this result in the virtual work equation, Eq. 1, gives

$$-T\,\delta y_B + mg\,\delta y_D = 0 \qquad \text{(Eq. 1 repeated)}$$

$3\delta y_D$, by Eq. 2

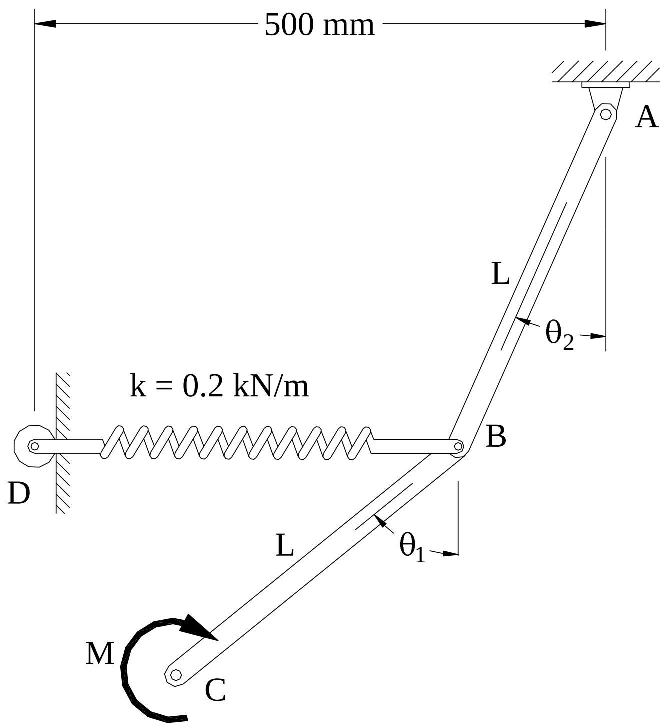
Thus

$$(-3T + mg)\,\delta y_D = 0$$

Dividing through by δy_D, substituting m = 90 kg, g = 9.81 m/s^2 and solving gives

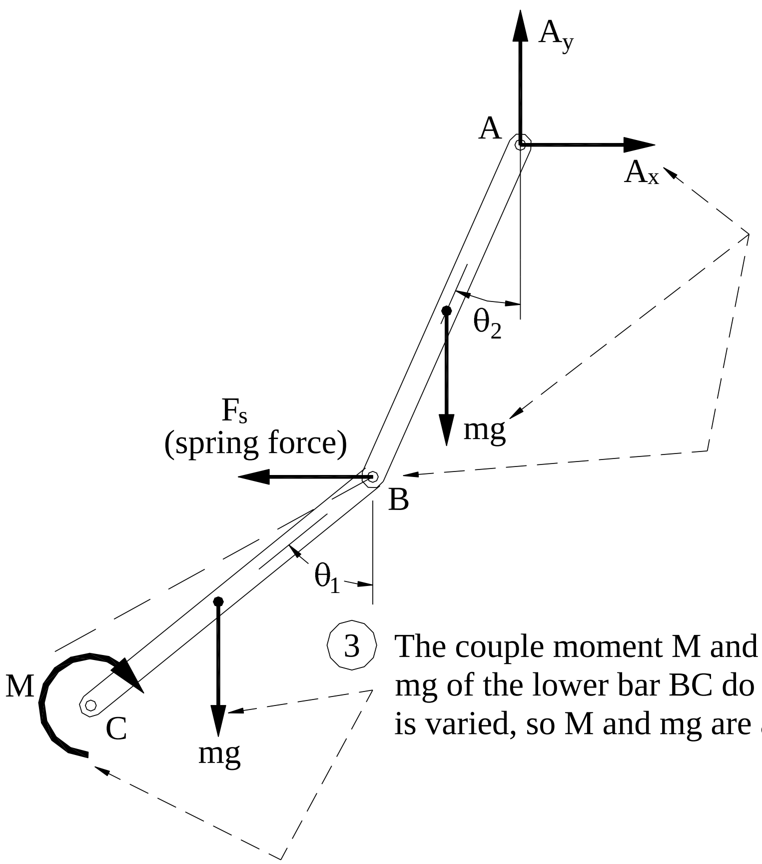
$$T = 294\text{ N} \qquad \leftarrow \text{Ans.}$$

17. Determine the equilibrium values of θ_1 and θ_2 for the two-bar linkage. The couple moment M = 5 N·m; each bar is uniform and has a mass m of 5 kg; the length L = 400 mm; and the unstretched length of the spring is 250 mm.



11.1 Virtual Work Example 17, page 2 of 9

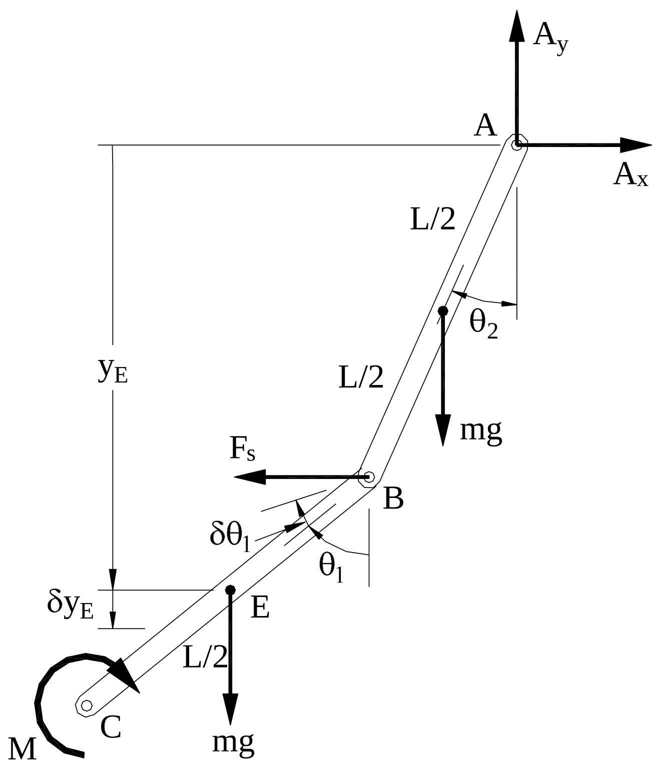
1 The system has two degrees of freedom because two coordinates θ_1 and θ_2 must be specified to define the position of the linkage. Consider a free-body diagram, and identify the active forces corresponding to a small change in θ_1—*while θ_2 is held fixed*.



2 Because point A does not move and θ_2 is fixed, the reactions A_x and A_y, the weight mg, and the spring force F_s do no work when θ_1 is varied a small amount. Thus they are not active forces.

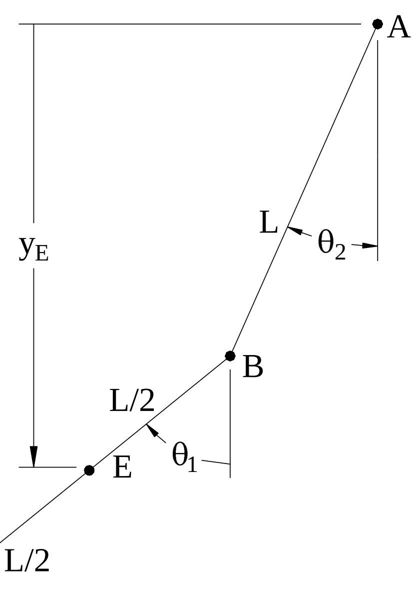
3 The couple moment M and the weight mg of the lower bar BC do work when θ_1 is varied, so M and mg are active forces.

4 In addition to the coordinate θ_1, introduce a vertical coordinate y_E measured downward from point A.



5 Compute the work done when the coordinates are increased a positive infinitesimal amount.

$$dU = 0:\ M\,\delta\theta_1 + mg\,\delta y_E = 0 \qquad (1)$$



6 Relate the differential δy_E to the angle change, $\delta\theta_1$, by writing

$$y_E = L\cos\theta_2 + (L/2)\cos\theta_1$$

and then differentiating with respect to θ_1, while holding θ_2 fixed. That is, take the partial derivative with respect to θ_1 to obtain

$$\delta y_E = -(L/2)\sin\theta_1\,\delta\theta_1 \qquad (2)$$

7 Substitute Eq. 2 for δy_E into the virtual-work equation:

$$M\,\delta\theta_1 + mg\,\delta y_E = 0 \qquad \text{(Eq. 1 repeated)}$$

$-(L/2)\sin\theta_1\,\delta\theta_1$, by Eq. 2

Thus

$$[M - (mgL/2)\sin\theta_1]\,\delta\theta_1 = 0$$

Since $\delta\theta_1 \neq 0$, it follows that

$$M - (mgL/2)\sin\theta_1 = 0 \qquad (4)$$

Substituting the following values into Eq. 4

$M = 5$ N·m $= 5\,000$ N·mm

$L = 400$ mm

$m = 5$ kg

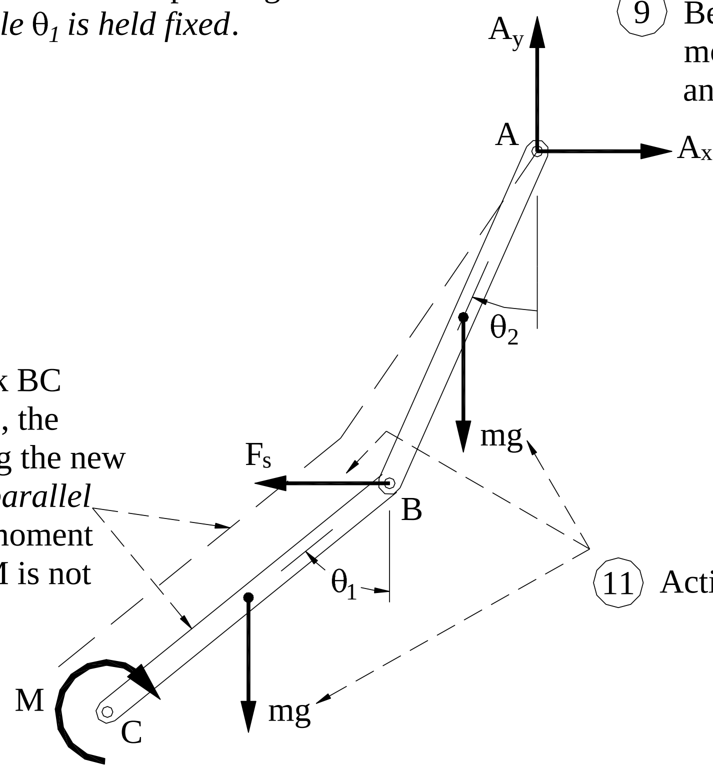
$g = 9.81$ kg·m/s^2

and solving gives

$\theta_1 = 30.6°$ ←Ans.

8 Next identify the active forces corresponding to a small change in θ_2 *while* θ_1 *is held fixed*.

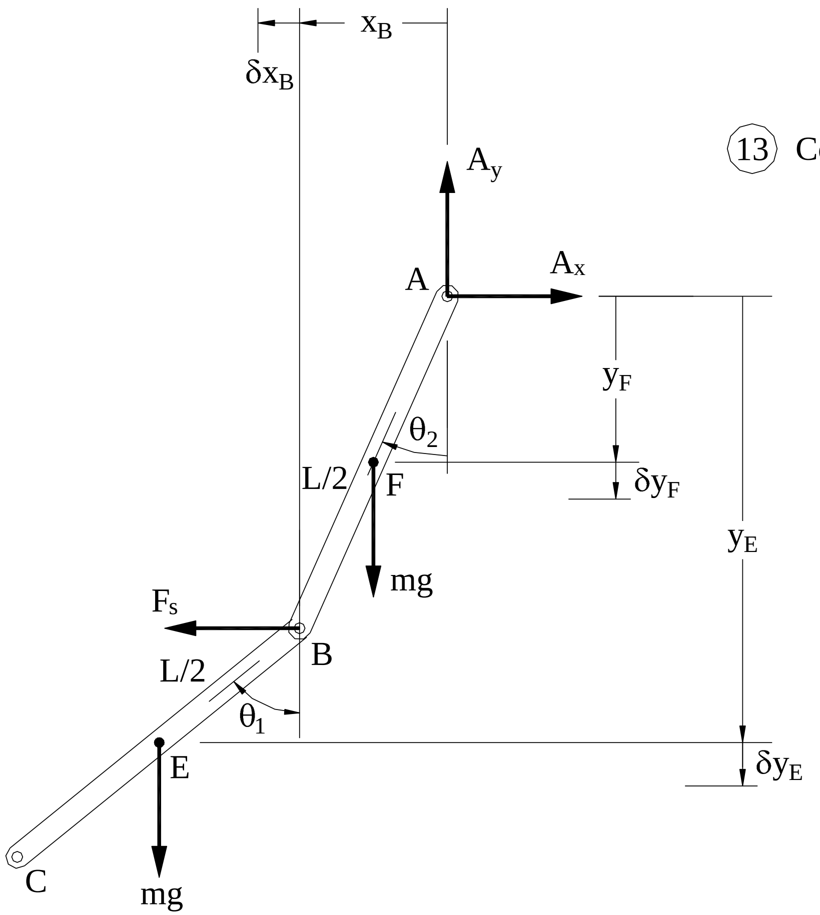
9 Because point A does not move, A_x and A_y do no work and thus are not active forces.



10 Because θ_1 is fixed, link BC does not rotate. That is, the dashed line representing the new position of link BC is *parallel* to BC. Hence couple moment M does no work, and M is not an active force.

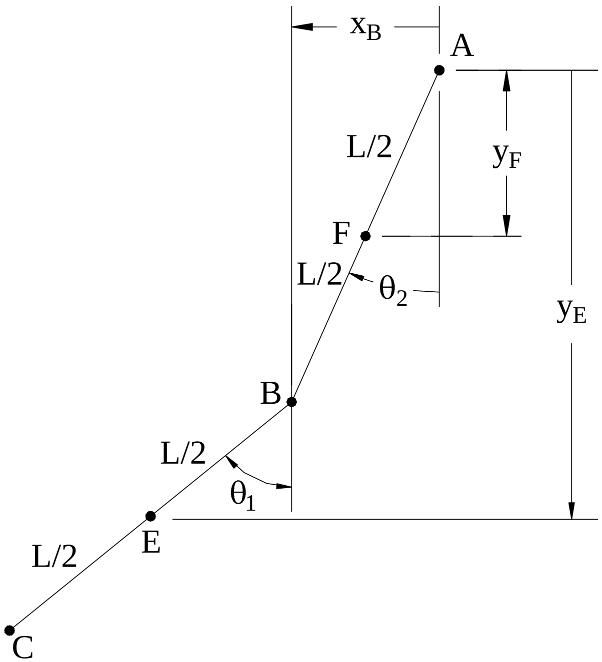
11 Active forces

12 Introduce coordinates measured from a fixed point to the point of application of the active forces.



13 Compute the work done.

$$\delta U = 0: \; mg\,\delta y_E + mg\,\delta y_F + F_s\,\delta x_B = 0 \tag{5}$$



14 Relate the differentials δy_E, δy_F, and δy_F to the change in angle $\delta\theta_2$:

$$x_B = L \sin \theta_2$$

$$y_F = (L/2) \cos \theta_2$$

$$y_E = L \cos \theta_2 + (L/2) \cos \theta_1$$

Differentiating each equation with respect to θ_2, with θ_1 held fixed, gives

$$\delta x_B = L \cos \theta_2 \, \delta\theta_2 \tag{6}$$

$$\delta y_F = -(L/2) \sin \theta_2 \, \delta\theta_2 \tag{7}$$

$$\delta y_E = -L \sin \theta_2 \, \delta\theta_2 \tag{8}$$

15 Substituting Eqs. 6-8 for the differentials into the virtual-work equation, Eq. 5, gives

$-L \sin\theta_2\, \delta\theta_2$, by Eq. 8

$$mg\,\delta y_F + mg\,\delta y_E + F_s\,\delta x_B = 0 \qquad \text{(Eq. 5 repeated)}$$

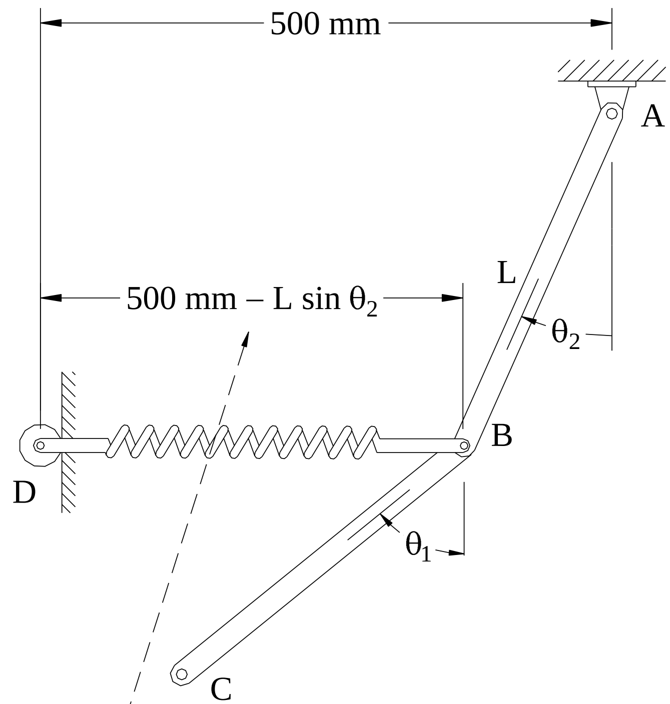
$-(L/2)\sin\theta_2\,\delta\theta_2$, by Eq. 7 $\quad L\cos\theta_2\,\delta\theta_2$, by Eq. 6

Thus

$$[-(3mg/2)\sin\theta_2 + F_s\cos\theta_2]\, L\,\delta\theta_2 = 0$$

Since $L\,\delta\theta_2 \neq 0$, it follows that

$$-(3mg/2)\sin\theta_2 + F_s\cos\theta_2 = 0 \qquad (9)$$



16 The force F_s in the spring can be related to θ_2:

$$\begin{aligned} F_s &= k \times \text{extension of spring} \\ &= k \times (\text{current length} - \text{unstretched length}) \\ &= k \times [(500\text{ mm} - L\sin\theta_2) - 250\text{ mm}] \\ &= k(250 - L\sin\theta_2) \qquad (10) \end{aligned}$$

11.1 Virtual Work Example 17, page 9 of 9

17 Substituting for F_s from Eq. 10 into the virtual-work equation, Eq. 9, gives

$$-(3mg/2)\sin\theta_2 + F_s\cos\theta_2 = 0 \qquad \text{(Eq. 9 repeated)}$$

$k(250 - L\sin\theta_2)$, by Eq. 10

or,

$$-(3mg/2)\sin\theta_2 + k(250 - L\sin\theta_2)\cos\theta_2 = 0 \qquad (11)$$

Substituting the following values into Eq. 11

$L = 400$ mm

$m = 5$ kg

$k = 0.2$ kN/m $= 0.2$ N/mm

$g = 9.81$ kg·m/s^2

and solving numerically gives

$$\theta_2 = 18.50° \qquad \leftarrow \text{Ans.}$$