11.1 Virtual Work
11.1 Virtual Work Example 1, page 1 of 5

1. Determine the force $P$ required to keep the two rods in equilibrium when the angle $\theta = 30^\circ$ and weight $W$ is 50 lb. The rods are each of length $L$ and of negligible weight. They are prevented from moving out of the plane of the figure by supports not shown.
The system has one degree of freedom, because specifying the value of a single coordinate, $\theta$, completely determines the configuration (shape) of the system. Consider a free-body diagram and identify the active forces—those forces that would do work if $\theta$ were increased slightly.

1. The force $W$ does work because point B moves up, so $W$ is an active force.

2. The force $P$ does work as point A moves to the right, so $P$ is an active force.

3. The reactions $C_x$ and $C_y$ do no work because point C does not move. Thus $C_x$ and $C_y$ are not active forces.

4. The normal force $N$ does no work because it is perpendicular to the displacement of point A. Thus $N$ is not an active force.

5. The reactions $C_x$ and $C_y$ do no work because point C does not move. Thus $C_x$ and $C_y$ are not active forces.
11.1 Virtual Work Example 1, page 3 of 5

6) Introduce coordinates measured from a fixed point, point C in the figure, to the point of application of the active forces.

7) Compute the work done when the coordinates are increased positive infinitesimal amounts, $\delta x_A$ and $\delta x_B$ (The custom followed by textbook writers is to use the Greek letter $\delta$ rather than simply writing $dx_A$ and $dy_B$ because the infinitesimals represent hypothetical motions—motions that are possible but are not necessarily motions that actually occur). The principle of virtual work says that the total work must add to zero for all possible motions, real or hypothetical—that is, "virtual."

$$\delta U = 0: \quad -P \delta x_A - W \delta y_B = 0 \quad (1)$$

A negative sign is present because the force P and displacement $\delta x_A$ are in opposite directions. That is, P does negative work (absorbs work from the system rather than adding work to the system). Similarly, the force W does negative work because W points down and $\delta y_B$ is directed up.
11.1 Virtual Work Example 1, page 4 of 5

Relate the differentials \( \delta x_A \) and \( \delta y_B \) through the change in the angle, \( \delta \theta \): From the figure, it follows that

\[
y_B = L \sin \theta
\]

(2)

To relate \( \delta y_B \) to \( \delta \theta \), use the ordinary formula from calculus for calculating a differential: if \( y = f(\theta) \), then the differential is

\[
dy = \frac{df}{d\theta} d\theta
\]

Applying this formula to Eq. 2 and using \( \delta \) rather than \( d \) gives

\[
\delta y_B = L \cos \theta \delta \theta
\]

(3)

Similarly

\[
x_A = 2L \cos \theta
\]

\[
\delta x_A = -2L \sin \theta \delta \theta
\]

(4)

Substitute Eqs. 3 and 4 for \( \delta y_B \) and \( \delta x_A \) into the virtual-work equation:

\[
-2L \sin \theta \delta \theta
\]

\[
-P \delta x_A - W \delta y_B = 0
\]

(Eq. 1 repeated)

\[
L \cos \theta \delta \theta
\]

or,

\[
(2P \sin \theta - W \cos \theta)(L \delta \theta) = 0
\]

Because \( L \delta \theta \neq 0 \), it follows that

\[
2P \sin \theta - W \cos \theta = 0
\]

Substituting the given values \( \theta = 30^\circ \) and \( W = 50 \) lb and solving gives

\[
P = 43.3 \text{ lb}
\]

\( \leftarrow \text{Ans.} \)
11.1 Virtual Work Example 1, page 5 of 5

10 Observation: the forces acting between the rods at pin B never occurred in the virtual-work equation because the work done by the equal-and-opposite force pairs acting between the parts of the body cancel out—for example the work done by \( B_x \) acting on rod AB has the opposite sign of the work done by \( B_x \) acting on rod BC. Forces such as \( B_x \) and \( B_y \) would have had to be considered if equilibrium equations rather than virtual work had been used.

11 Often virtual work is easier to use than equilibrium equations for problems involving connected rigid bodies (typically machines and mechanisms), but this advantage exists only if the relation between displacements can be found easily. If the geometry is difficult, then using equilibrium equations is probably the better approach.
11.1 Virtual Work Example 2, page 1 of 3

2. Determine the value of moment M required to maintain the mechanism in the position shown, if $\theta = 35^\circ$ and $W = 200$ lb.
The system has one degree of freedom: specifying the value of the single coordinate, $\theta$, completely determines the configuration of the system. Consider a free-body diagram and identify the active forces.

The reactions at A and C do no work because points A and C do not move. Thus the reactions are not active forces.

The weight W of the block does work because the center of gravity of the block moves vertically; thus the weight is an active force.

Couple-moment M does work because member CD rotates, so M is an active "force" (better said, "an active moment" or "active generalized force").
11.1 Virtual Work Example 2, page 3 of 3

Introduce a coordinate \( y \) measured from the fixed point \( A \) to the point of application of the force \( W \).

Compute the work done when \( y \) and \( \theta \) are increased a positive infinitesimal amount.

\[
\delta U = -M \delta \theta + W \delta y = 0 \tag{1}
\]

Note that the work done by a moment equals moment times angle of rotation. Here the work is negative because \( M \) and \( \delta \theta \) have opposite senses.

Next use geometry to relate the \( y \) and \( \theta \):

\[
y = (2 \text{ ft}) \sin \theta \tag{2}
\]

Differentiating gives

\[
\delta y = 2 \cos \theta \delta \theta \tag{2}
\]

Substitute Eq. 2 for \( \delta y \) into Eq. 1.

\[
-M \delta \theta + W \delta y = 0 \tag{Eq. 1 repeated}
\]

\[
2 \cos \theta \delta \theta
\]

Thus

\[
(-M + 2W \cos \theta) \delta \theta = 0 \tag{3}
\]

Substituting the given values \( W = 200 \text{ lb} \) and \( \theta = 35^\circ \) into Eq. 3 and noting \( \delta \theta \neq 0 \) gives

\[
-M + 2(200 \text{ lb}) \cos 35^\circ = 0
\]

Solving gives

\[
M = 328 \text{ lb-ft} \quad \leftarrow \text{Ans.}
\]
3. Determine the value of the weight $W$ required to maintain the mechanism in the position shown, if $P = 50$ N.
11.1 Virtual Work Example 3, page 2 of 4

1. The system has one degree of freedom because if the displacement of one end of a bar is known, the displacement of the other bars can be found by similar triangles (as will be shown below). Consider a free-body diagram and identify the active forces corresponding to a small change in configuration of the system.

Free-body diagram (The dashed line shows the position of the system after the bars have been displaced a small amount.)

2. The force W does work if A moves vertically, so W is an active force.

3. The force P does work as point F moves vertically, so P is an active force.

4. The reaction forces at B and E do no work because B and E do not move; thus the reactions are not active forces.
11.1 Virtual Work Example 3, page 3 of 4

(5) Introduce coordinates measured from a fixed point to the point of application of the active forces.

\[ U = W \delta y_A + P \delta y_F = 0 \]  
(1)

The force \( W \) does negative work because it is directed down, while the displacement is up.

(6) Compute the work done when the coordinates are increased a positive infinitesimal amount.

(7) Now relate the differentials \( \delta y_A \) and \( \delta y_F \). By similar triangles

\[ \frac{\delta y_D}{2} = \frac{\delta y_F}{3} \]  
(2)

and

\[ -\frac{\delta y_A}{3} = \frac{\delta y_C}{3} \]  
(3)

Member DC does not change length so ends C and D move down the same amount, that is,

\[ \delta y_C = \delta y_D \]  
(4)

Eqs. 2, 3, and 4 imply

\[ \delta y_A = \frac{2}{3} \delta y_F \]  
(5)
Substitute Eq. 5 into the virtual-work equation, Eq. 1:

\[ -W \delta y_A + P \delta y_F = 0 \]  
(Eq. 1 repeated)

\[ (2/3) \delta y_F \]

Thus

\[ [-W + P] \delta y_F = 0 \]

or, since \( \delta y_F \neq 0 \) and \( P \) is given as 50 N,

\[ -W + 50 = 0 \]

Solving gives

\[ W = 75 \text{ N} \]

Ans.
4. Determine the force Q necessary to maintain equilibrium when force $P = 400$ N.
The system has one degree of freedom because if member ABC is rotated about point B a small amount, then the position of CD and DEFG can be determined. Consider a free-body diagram and identify the active forces corresponding to the displacements shown.

1. The system has one degree of freedom because if member ABC is rotated about point B a small amount, then the position of CD and DEFG can be determined. Consider a free-body diagram and identify the active forces corresponding to the displacements shown.

2. P does work so is an active force.

3. Points B and E do not move, so the reactions at these points do no work.

4. Q does work so is an active force.
Introduce coordinates measured from a fixed point to the point of application of the forces.

Compute the work done when the coordinates are increased a positive infinitesimal amount.

$$\delta U = 0: P \delta x_A + Q \delta y_G = 0$$ (1)
11.1 Virtual Work Example 4, page 4 of 7

Relate the differentials $\delta x_A$ and $\delta y_G$. Begin by noting that because $\delta \theta_A$ is a small angle, the tangent of $\delta \theta_A$ can be replaced by the angle itself:

$$\delta \theta_A = \frac{\delta x_A}{300 \text{ mm}} \tag{2}$$

Member ABC is a rigid body, and all parts must rotate the same amount. Thus

$$\delta \theta_C = \delta \theta_A$$

Substituting for $\delta \theta_A$ from Eq. 2 then gives

$$\delta \theta_C = \frac{\delta x_A}{300 \text{ mm}}$$

Again using the small angle approximation for the tangent gives

$$\delta y_C = (250 \text{ mm}) \delta \theta_C$$

$$= (250 \text{ mm}) \left( \frac{\delta x_A}{300 \text{ mm}} \right)$$

$$= (5/6) \delta x_A$$

Member CD is a rigid body and thus doesn't shorten or lengthen. It follows that

$$\delta y_D = \delta y_C$$

Thus

$$\delta y_D = (5/6) \delta x_A \tag{3}$$
11.1 Virtual Work Example 4, page 5 of 7

Using the small angle approximation for the tangent gives

\[ \delta \theta_D = \frac{\delta y_D}{250 \text{ mm}} = \frac{\delta x_A}{300 \text{ mm}} \]

[(5/6) \delta x_A], by Eq. 3

Member DEFG is a rigid body and so all parts must rotate the same amount. Thus

\[ \delta \theta_F = \delta \theta_D \]

Thus

\[ \delta \theta_F = \frac{\delta x_A}{300 \text{ mm}} \] (4)
11.1 Virtual Work Example 4, page 6 of 7

The remaining step is to relate $\delta y_G$ to $\delta \theta_F$. We can do this in two ways, by geometry or by calculus. Let’s begin with the geometric approach. First consider a rotation of line EG by an amount $\delta \theta_F$.

Length $= \sqrt{(300 \text{ mm})^2 + (400 \text{ mm})^2} = 500 \text{ mm}$

So, considering the small triangle gives

\[ |\delta y_G| = (500 \delta \theta_F) \sin \phi \]

or,

\[ \delta y_G = -\frac{4}{3} \delta x_A \]

(Insert a minus sign, because $\delta y_G$ was originally defined as positive down)
11.1 Virtual Work Example 4, page 7 of 7

17) Consider an alternative solution for $\delta y_G$, based on calculus:

$$\begin{align*}
\phi &= (500 \text{ mm}) \cos \phi \\
\delta y_G &= -500 \sin \phi \, \delta \phi \\
\phi &= \delta \theta_F \text{ because both angles measure the rotation of line EG} \\
\delta y_G &= -500 \left( \frac{400 \text{ mm}}{500 \text{ mm}} \right) \delta \theta_F \\
&= -\frac{400 \delta x_A}{300}, \text{ by Eq. 4} \\
&= -\frac{4 \delta x_A}{3} \text{ (Same as Eq. 5)}
\end{align*}$$

19) Note that we can't write $y_G = 300 \text{ mm}$ and then differentiate to get $\delta y_G$ (which would give $\delta y_G = 0$). The equation for $y_G$ must define a continuous and differentiable function, not a relationship that is only valid at a single value of $\phi$.

20) Substitute for $\delta y_G$ from Eq. 5 into the virtual work equation:

$$P \delta x_A + Q \delta y_G = 0 \quad \text{(Eq. 1 repeated)}$$

or,

$$-\frac{4}{3} \delta x_A, \text{ by Eq. 5}$$

$$[P + (-4/3)Q] \delta x_A = 0$$

Dividing through by $\delta x_A$ and using the given value $P = 400 \text{ N}$ yields

$$Q = 300 \text{ N} \quad \leftarrow \text{Ans.}$$
5. Link AB is connected to collar A, which can slide with negligible friction on horizontal rod EF. Determine the value of force Q necessary to maintain equilibrium when $\theta = 50^\circ$, $L = 300$ mm, and $P = 100$ N.
Forces from pin D are not active forces.

The force from the rod acting on the collar is not an active force.

Free-body diagram showing active forces corresponding to a small increase in $\theta$. 

Forces from pin D are not active forces.
11.1 Virtual Work Example 5, page 3 of 4

2. Introduce coordinates measured from the fixed point D to the point of application of the active forces.

3. Compute the work done when the coordinates are increased a positive infinitesimal amount.

\[ \delta U = 0: \quad Q \delta x_A + P \delta y_C = 0 \quad (1) \]

4. Relate the differential \( \delta x_A \) to the change in angle, \( \delta \theta \).

5. \[ x_A = L \cos \theta + \frac{L}{2} + \frac{L}{2} \]
\[ \delta x_A = -L \sin \theta \delta \theta + \frac{\delta L}{2} + \frac{\delta L}{2} \quad (2) \]

(Length \( L \) does not change)
Relate the differential $\delta y_C$ to the change in angle $\delta \theta$. From the figure, we have

$$y_B = L \sin \theta$$

$$\delta y_B = L \cos \theta \delta \theta$$

By similar triangles,

$$\frac{\delta y_B}{L/2 + L/2} = \frac{\delta y_C}{L/2}$$

Thus

$$\delta y_C = \frac{L \cos \theta \delta \theta}{2} \quad (3)$$

Substitute Eqs. 2 and 3 for $\delta x_A$ and $\delta y_C$ into the virtual work equation, Eq. 1:

$$-L \sin \theta \delta \theta, \text{ by Eq. 2}$$

$$Q \delta x_A + P \delta y_C = 0 \quad \text{(Eq. 1 repeated)}$$

$$\frac{L \cos \theta \delta \theta}{2}, \text{ by Eq. 3}$$

Thus

$$(-Q \sin \theta + P \frac{\cos \theta}{2})(L \delta \theta) = 0$$

Because $L \delta \theta \neq 0$, it follow that

$$-Q \sin \theta + P \frac{\cos \theta}{2} = 0$$

Substituting $\theta = 50^\circ$ and $P = 100 \text{ N}$ and solving for $Q$ gives

$$Q = 42.0 \text{ N} \quad \leftarrow \text{Ans.}$$
11.1 Virtual Work Example 6, page 1 of 5

6. Rotating the threaded rod AC of the automobile jack causes joints A and C to move closer together, thus raising the weight \( W \). Determine the axial force in the rod, if \( \theta = 30^\circ \) and \( W = 2 \) kN.
11.1 Virtual Work Example 6, page 2 of 5

1) The system has one degree of freedom because once $\theta$ is specified, the location of all parts of the jack can be determined. Consider a free-body diagram of the jack and identify the active forces corresponding to a small change in $\theta$.

Free-body diagram

2) To get a virtual-work equation that contains the axial force in the rod, it is necessary to exclude the rod from the free-body diagram. The effect of the rod is then represented by the two forces $F_r$. $W$ and the two $F_r$ forces are the active forces.
11.1 Virtual Work Example 6, page 3 of 5

3) Introduce coordinates measured from fixed points to the points of application of the active forces.

4) Calculate the work done.

\[ \delta U = 0: \quad -W \delta y_B - F_r \delta x_A - F_r \delta x_C = 0 \] (1)
11.1 Virtual Work Example 6, page 4 of 5

5) Relate $\delta y_B$, $\delta x_A$, and $\delta x_C$ through the angle change, $d\theta$.

6) Distance "a," from the intersection of the two sloping members, point E, to point B, does not change as $\theta$ changes. Thus when "a" is differentiated, the result is zero: $\delta a = 0$.

7) $x_A = (150 \text{ mm}) \cos \theta$

$\delta x_A = -150 \sin \theta \delta \theta \quad (2)$

$x_C = (150 \text{ mm}) \cos \theta$

$\delta x_C = -150 \sin \theta \delta \theta \quad (3)$

$y_B = 2(150 \text{ mm}) \sin \theta + a$

$\delta y_B = 300 \cos \theta \delta \theta + \delta a \delta \theta \quad (4)$
Substituting the expressions for $\delta y_B$, $\delta x_A$, and $\delta x_C$ and into the virtual work equation, Eq. 1, gives

$$-W \delta y_B - F_r \delta x_A - F_r \delta x_C = 0$$

(Eq. 1 repeated)

or,

$$[ -300W \cos \theta + 2(150)F_r \sin \theta ] \delta \theta = 0$$

Dividing through by $\delta \theta$, substituting the given values $W = 2$ kN and $\theta = 30^\circ$, and solving gives

$$F_r = 3.46 \text{ kN} \quad \leftarrow \text{Ans.}$$
7. The original length of the spring is $L$. Determine the angle $\theta$ for equilibrium if $L = 3$ m and $P = 300$ N.

Spring constant, $k = 200$ N/m
The spring is not part of the free-body; the effect of the spring is represented by the forces $F_s$.

Forces $P$ and $F_s$ are active forces because their points of application move in the direction of the forces.

The system can be described by a single coordinate, $\theta$. Consider a free-body diagram and identify the active forces corresponding to a small change in $\theta$.

Reactions $A_y$ and $B_y$ do no work because they are perpendicular to the displacement of points A and B.

The spring is not part of the free-body; the effect of the spring is represented by the forces $F_s$. 
11.1 Virtual Work Example 7, page 3 of 5

5) Introduce coordinates measured from the point O directly above pin C to the point of application of the active forces.

6) Compute the work done when the coordinates are increased a positive infinitesimal amount.

\[
\delta U = P \delta y_F + F_s \delta x_D + F_s \delta x_E = 0 \tag{1}
\]
Relate the differentials $\delta y_F$, $\delta x_D$, and $\delta x_E$ to the angle change $\delta \theta$. From the figure, we see that

- $y_F = 3L \sin \theta$
- $x_E = L \cos \theta$
- $x_D = L \cos \theta$

Differentiating gives

- $\delta y_F = 3L \cos \theta \, \delta \theta$ (2)
- $\delta x_E = -L \sin \theta \, \delta \theta$ (3)
- $\delta x_D = -L \sin \theta \, \delta \theta$ (4)

We can use the same figure to calculate the length of the spring, $L'$, say:

- $L' = \text{distance } DE$
  - $= 2L \cos \theta$ (5)
8 The force in the spring is, then,

\[ F_s = k \times \text{compression of spring} \]

\[ = k \times (\text{original length} - \text{final length}) \]

\[ = kL(1 - 2 \cos \theta) \quad (6) \]

Substitute from Eqs. 2, 3, 4, and 6 into the virtual work equation, Eq. 1:

\[ P \delta y_F + F_s \delta x_D + F_s \delta x_E = 0 \quad \text{(Eq. 1 repeated)} \]

\[ 3L \cos \theta \delta \theta \text{ by Eq. 2} \]
\[ -L \sin \theta \delta \theta \text{ by Eq. 3} \]
\[ -L \sin \theta \delta \theta \text{ by Eq. 4} \]

Thus

\[ [3P \cos \theta - 2kL(1 - 2 \cos \theta) \sin \theta](L \delta \theta) = 0 \]

or, since \( L \delta \theta \neq 0 \),

\[ [3P \cos \theta - 2kL(1 - 2 \cos \theta) \sin \theta] = 0 \quad (7) \]

9 Substituting the given values \( P = 300 \text{ N}, \)
\( k = 200 \text{ N/m}, \) and \( L = 3 \text{ m} \) into Eq. 7 and solving numerically gives

\[ \theta = 69.1^\circ \quad \leftarrow \text{Ans.} \]
8. Collars A and B can slide freely on rods CD and CE. Determine the values of $x$ and $y$, given that forces $P = 900$ N and $Q = 800$ N. The unstretched length of the spring is 0.2 m, and the weight of the collars is negligible.
The system has two degree of freedom since both $x$ and $y$ coordinates must be known if the configuration of the system is to be determined. Consider a free-body diagram and identify the active forces corresponding to a small change in $x$, while $y$ is held fixed.

Because $y$ is fixed, collar B does not move, and so none of the forces acting on B is an active force.

The forces acting on collar A, $F_s$ (the spring force) and the 900-N force, are the only active forces.
11.1 Virtual Work Example 8, page 3 of 5

The coordinate \( x \) locates the position of the 700-N force. Introduce an additional coordinate, \( L \), that locates the point of application of the spring force, \( F_s \).

Compute the work done:

\[
\delta U = 0: (900 \text{ N}) \delta x - F_s \delta L = 0 \quad (1)
\]

Relate \( \delta x \) and \( \delta L \):

\[
L^2 = x^2 + y^2 \quad (2)
\]

Differentiating gives

\[
2L \delta L = 2x \delta x + 2y \delta y
\]

Because \( y \) is fixed

Thus

\[
\delta L = \frac{x}{L} \delta x
\]

Introduce the latter equation into Eq. 1:

\[
(900 \text{ N}) \delta x - F_s \left( \frac{x}{L} \right) \delta x = 0 \quad (\text{Eq. 1 repeated})
\]

Thus

\[
(900 - F_s \frac{x}{L}) \delta x = 0
\]

Dividing through by \( \delta x \) and re-arranging gives

\[
F_s x = 900L \quad (3)
\]
Next, hold x fixed and compute the work done when collar B moves an amount $\delta y$. Following the same steps as were used for $\delta x$ leads to

$$F_y = 800L$$  \hspace{1cm} (4)

The spring force, $F_s$, is related to $L$:

$$F_s = k \times \text{extension of the spring}$$

$$= (9000 \text{ N/m}) \times (L - \text{original length})$$

Thus

$$F_s = 9000L - 1800$$  \hspace{1cm} (5)

We now have four simultaneous nonlinear equations to solve:

$$L^2 = x^2 + y^2$$  \hspace{1cm} (2)

$$F_s x = 900L$$  \hspace{1cm} (3)

$$F_s y = 800L$$  \hspace{1cm} (4)

$$F_s = 9000L - 1800$$  \hspace{1cm} (5)
11.1 Virtual Work Example 8, page 5 of 5

These equations can be solved directly if a calculator that is able to handle such systems is available.

Alternatively, proceed as follows: square both sides of Eqs. 3 and 4 and add the results to get

\[(F_x)^2 + (F_y)^2 = (900L)^2 + (800L)^2.\]

or

\[F_s^2(x^2 + y^2) = L^2(900^2 + 800^2)\]

L, by Eq. 2

Solving gives

\[F_s = 1204 \text{ N}\]

Using this result in Eq. 5 gives

\[F_s = 9000L - 1800\]

1204 N

Solving gives

\[L = 0.3338 \text{ m}\]

Distance x can now be found from Eq. 3:

\[F_s x = 900L\]

1204 N 0.3338 m

Solving gives

\[x = 0.250 \text{ m}\]

→ Ans.

Distance y can be found from Eq. 4:

\[F_s y = 800L\]

1204 N 0.3338 m

Solving gives

\[y = 0.222 \text{ m}\]

→ Ans.
**11.1 Virtual Work Example 9, page 1 of 4**

9. Determine the moment \( M \) applied to the crankshaft that will keep the piston motionless when a pressure \( p = 400 \text{ psi} \) acts on the top of the piston and \( \theta = 25^\circ \). The diameter of the piston is 3 in., and the piston slides with negligible friction in the cylinder.

1. The system can be described by a single coordinate, \( \theta \). Consider a free-body diagram and identify the active forces corresponding a small change in \( \theta \).

2. The resultant of the pressure is an active force:

\[
(400 \text{ psi})(\pi)(3 \text{ in.}/2)^2 = 2827 \text{ lb}
\]

3. Since friction is negligible, only the normal force \( N \) acts on the side of the piston. The normal force does no work since it acts perpendicular to the motion of the piston.

4. The reaction forces \( A_x \) and \( A_y \) do no work, because point A does not move. The moment \( M \) does work as link AB rotates.
11.1 Virtual Work Example 9, page 2 of 4

5) Introduce coordinates measured from a fixed reference at point A.

6) Compute the work done when the coordinates are increased a positive infinitesimal amount.

\[\delta U = -(2827 \text{ lb}) \, \delta y_C - M \, \delta \theta = 0 \]
11.1 Virtual Work Example 9, page 3 of 4

7. Relate $\delta y_C$ to $\delta \theta$:

$$y_C = (4 \text{ in.}) \cos \theta + (9 \text{ in.}) \cos \phi + a$$

$$\delta y_C = -4 \sin \theta \delta \theta - 9 \sin \phi \delta \phi + \delta a$$

(2)

8. Relate $\theta$ to $\phi$ by the law of sines,

$$\frac{\sin \phi}{4 \text{ in.}} = \frac{\sin \theta}{9 \text{ in.}}$$

(3)

Distance "a" does not change as $\theta$ is changed so $\delta a = 0$.

Differentiating gives

$$\frac{\cos \phi \delta \phi}{4} = \frac{\cos \theta \delta \theta}{9}$$

Thus

$$\delta \phi = \frac{4 \cos \theta \delta \theta}{9 \cos \phi}$$

(4)
11.1 Virtual Work Example 9, page 4 of 4

Using Eq. 4 in Eq. 2 gives
\[ \delta y_C = -4 \sin \theta \delta \theta - 9 \sin \phi \delta \phi \]  
(Eq. 2 repeated)
\[
\frac{4 \cos \theta}{9 \cos \phi} \delta \theta, \text{ by Eq. 4}
\]
\[ = (-4 \sin \theta - 4 \tan \phi \cos \theta) \delta \theta \]  
(Eq. 5)

Substituting Eq. 5 in the virtual work equation gives
\[ -(2827) \delta y_C - M \delta \theta = 0 \]  
(Eq. 1 repeated)
\[
(-4 \sin \theta - 4 \tan \phi \cos \theta) \delta \theta, \text{ by Eq. 5}
\]
or
\[ [4(2827)(\sin \theta + \tan \phi \cos \theta) - M] \delta \theta = 0 \]

Dividing through by \( \delta \theta \) and solving gives
\[ M = 4(2827)(\sin \theta + \tan \phi \cos \theta) \]  
(Eq. 6)

Substituting the given value \( \theta = 25^\circ \) into Eq. 3 yields
\[ \frac{\sin \phi}{4 \text{ in.}} = \frac{\sin \theta}{9 \text{ in.}} \]  
(Eq. 3 repeated)

which can be solved to give \( \phi = 10.83^\circ \).
11.1 Virtual Work Example 10, page 1 of 5

10. Pin B is rigidly attached to member AC and moves in the smooth quarter-circle slot EF. Determine the value of force Q necessary to keep the system in equilibrium, if $\theta = 30^\circ$, $L = 400$ mm, $a = 120$ mm, and $P = 200$ N.
1. The system configuration can be defined by the single coordinate, \( \theta \). Consider a free-body diagram showing the active forces corresponding to a small increase in \( \theta \).

2. Pin B must move in the slot, that is, in a direction tangent to the quarter circle and thus perpendicular to radius DB. Thus the force R from the slot does no work because R is perpendicular to the motion of the pin.

3. Point A must move horizontally. It is difficult to tell if A moves to the right or left, but fortunately it makes no difference. The important thing is to note that force \( A_y \) from the rollers does no work so is not an active force.

4. Point C moves both horizontally and vertically. It is difficult to tell if C moves vertically up or vertically down, but it makes no difference. All we need to note is that force P is an active force.
5) Introduce coordinates measured from the fixed point D to the point of application of the active forces P and Q.

6) Compute the work done when the coordinate are increased a positive infinitesimal amount.

\[ \delta U = 0: \quad -Q \delta x_A + P \delta y_C = 0 \]
Relate the differentials $\delta x_A$ and $\delta y_C$ to the angles $\theta$ and $\phi$. Begin with $y_C$.

\[ y_C = L \sin \theta \]
\[ \delta y_C = L \cos \theta \, \delta \theta \]  

Note that this equation shows $\delta y_C$ is positive if $\delta \theta$ is positive, that is, point $C$ moves up as $\theta$ increases.

Relate $\theta$ and $\phi$ through the law of sines:

\[ \frac{\sin (180^\circ - \phi)}{L/2} = \frac{\sin \theta}{a} \]

Because $\sin (180^\circ - \phi) = \sin \phi$, the last equation can be written as

\[ a \sin \phi = (L/2) \sin \theta \]  

Differentiating gives

\[ a \cos \phi \, \delta \phi = (L/2) \cos \theta \, \delta \theta \]

so

\[ \delta \phi = \frac{L \cos \theta \, \delta \theta}{2a \cos \phi} \]  

Using Eq. 5 in Eq. 3 gives

\[ \delta x_A = -(L/2) \sin \theta \, \delta \theta + a \sin \phi \, \delta \phi \]  

(Eq. 3 repeated)

\[ = (-\sin \theta + \cos \theta \tan \phi)(L/2) \, \delta \theta \]  

(Eq. 6)
The angle $\phi$ in Eq. 6 can be calculated by substituting the given values $\theta = 30^\circ$, $L = 400$ mm, and $a = 120$ mm into Eq. 4:

$$a \sin \phi = \frac{L}{2} \sin \theta \quad \text{(Eq. 4 repeated)}$$

and solving to get $\phi = 56.44^\circ$.

Although it is not necessary for solving the problem, we can now determine whether point $A$ moves to the left or to the right. From Eq. 6 we have

$$\delta x_A = (-\sin \theta + \cos \theta \tan \phi)(\frac{L}{2}) \delta \theta \quad \text{(Eq. 6 repeated)}$$

Substituting $\theta = 30^\circ$ and $\phi = 56.44^\circ$ into this equation gives

$$\delta x_A = (0.8054)(\frac{L}{2}) \delta \theta \quad \text{(7)}$$

That is, $\delta x_A$ is positive when $\delta \theta$ is positive, so $x_A$ increases, that is, point $A$ moves to the left for the particular values of $\theta$, $a$, and $L$ of our problem.

Substituting Eqs. 2 and 7 for $\delta y_C$ and $\delta x_A$ into the virtual-work equation, Eq. 1, gives

$$-Q \delta x_A + P \delta y_C = 0 \quad \text{(Eq. 1 repeated)}$$

$$L \cos \theta \delta \theta, \text{ by Eq. 2}$$

so

$$[-Q(0.8054)/2 + P \cos \theta](L \delta \theta) = 0$$

Because $L \delta \theta \neq 0$, it follows that

$$-Q(0.8054)/2 + P \cos \theta = 0$$

Substituting $\theta = 30^\circ$ and $P = 200$ N and solving gives

$$Q = 430 \text{ N} \quad \text{←Ans.}$$
11. A scissors lift is used to raise a weight $W = 800$ lb. Determine the force exerted on pin $F$ by the hydraulic cylinder $AF$ when $\theta = 35^\circ$. Each linkage member is 2-ft long and pin connected at its midpoint and endpoints. The lift consists of two identical linkages and cylinders—the one shown and one directly behind it.
Consider a free-body diagram and identify the active forces associated with a small change in $\theta$.

1. Each side of the lift carries half of the load. The W/2 load does work, so it is an active force.

2. The force $F_{FA}$ of the hydraulic cylinder acting on pin $F$ does work as pin $F$ moves.

3. The force $F_{FA}$ of the hydraulic cylinder acting on pin $A$ does no work because pin $A$ does not move. For the same reason, the reaction forces $A_x$ and $A_y$ from the support do no work.

4. The reaction force at $B$ does no work because it is vertical while the motion of point $B$ is horizontal.
Introduce coordinates $y_J$ and $s_F$ measured from the fixed point $A$ to the point of application of the active forces.

Compute the work done when the coordinates are increased a positive infinitesimal amount:

$$\delta U = -(W/2) \delta y_J + F_{FA} \delta s_F = 0$$  \hspace{1cm} (1)
11.1 Virtual Work Example 11, page 4 of 5

7) Relate the coordinate $y_J$ to the angle $\theta$:

$$y_J = (6 \text{ ft}) \sin \theta$$

$$\delta y_J = 6 \cos \theta \ \delta \theta \quad (2)$$

8) To relate $s_F$ to $\theta$, consider triangle AFCO.

9) Use the Pythagorean Theorem and then differentiate to get $\delta s_F$.

\[
s_F = \sqrt{(3 \sin \theta)^2 + (1 \cos \theta)^2}
\]

\[
\delta s_F = \frac{1}{2} \frac{2(3 \sin \theta)(3 \cos \theta) \delta \theta + 2(\cos \theta)(-\sin \theta) \delta \theta}{\sqrt{(3 \sin \theta)^2 + (1 \cos \theta)^2}}
\]

\[
= \frac{8 \sin \theta \cos \theta \delta \theta}{\sqrt{9 \sin^2 \theta + \cos^2 \theta}} \quad (3)
\]
Substituting for $sF$ and $yJ$ from Eqs. 2 and 3 in the virtual-work equation, Eq. 1, gives

$$\left(\frac{W}{2}\right) yJ + FFA sF = 0 \quad (\text{Eq. 1 repeated})$$

Thus

$$3W + 8F_{FA} \cos \theta = 0$$

This implies, since $\cos \theta \neq 0$, that

$$-3W + \frac{8F_{FA} \sin \theta}{\sqrt{9 \sin^2 \theta + \cos^2 \theta}} = 0$$

Substituting the given values $\theta = 35^\circ$ and $W = 800$ lb and solving gives

$$F_{FA} = 997 \text{ lb} \quad \leftarrow \text{Ans.}$$
11.1 Virtual Work Example 12, page 1 of 5

12. The unstretched length of the spring is 1 m. Determine the value of $\theta$ for equilibrium when force $P = 2 \text{kN}$.

The system can be described by a single coordinate, $\theta$. Consider a free-body diagram and identify the active forces corresponding to a small change in $\theta$.

1. The spring is not part of the free-body; the effect of the spring is represented by the force $F_s$, which is an active force.

2. Force $P$ is an active force.

Free-body diagram
11.1 Virtual Work Example 12, page 2 of 5

4) Introduce coordinates measured from the fixed points A and B to the point of application of the forces.

5) Compute the work done when the coordinates are increased a positive infinitesimal amount.

\[ \delta U = P \delta y_D - F_s \delta s_C = 0 \] (1)
11.1 Virtual Work Example 12, page 3 of 5

6) Relate the differentials $\delta s_C$ and $\delta y_D$ to the angle change $\delta \theta$.

$$ s_C^2 = (2 \text{ m})^2 + (s_B)^2 - 2(2 \text{ m})(s_B) \cos (\phi + \theta) \quad (3) $$

Here

$$ s_B = \sqrt{(3 \text{ m})^2 + (1.5 \text{ m})^2} = 3.354 \text{ m} \quad (4) $$

and

$$ \phi = \tan^{-1} \frac{1.5 \text{ m}}{3 \text{ m}} = 26.565^\circ \quad (5) $$

To avoid having to write equations containing several four and five-digit numbers, introduce intermediate variables $a$ and $b$: 

$$ s_C^2 = a - b \cos (\phi + \theta) \quad (Eq. 3 \text{ repeated}) $$

Thus

$$ s_C^2 = a - b \cos (\phi + \theta) \quad (6) $$

where

$$ a = 2^2 + (s_B)^2 = 4 + 3.354^2 = 15.249 \text{ m by Eq. 4} $$

7) $y_D = (2 \text{ m} + 4 \text{ m}) \sin \theta$

$$ \delta y_D = 6 \cos \theta \delta \theta \quad (2) $$

8) Law of cosines applied to triangle ABC:
11.1 Virtual Work Example 12, page 4 of 5

9. The parameter $b$ can also be evaluated, for later use:

$$b = 2(2)(s_B)$$

$$= 4(3.354)$$

$$= 13.416 \text{ m}$$ (8)

$s_C$ can be related to $\theta$ by differentiating Eq. 6:

$$s_C^2 = a - b \cos (\phi + \theta) \quad (\text{Eq. 6 repeated})$$

$$2s_C \delta s_C = b \sin (\phi + \theta) \delta \theta$$

so

$$\delta s_C = \frac{b \sin (\phi + \theta) \delta \theta}{2s_C} \quad (9)$$

Taking the square root of both sides of Eq. 6 gives an equation for $s_C$.

$$s_C = \sqrt{a - b \cos (\phi + \theta)} \quad (10)$$

10. The spring force $F_s$ can be expressed in terms of $s_C$:

$$F_s = k \times (\text{extension of the spring})$$

$$= k \times (\text{stretched length} - \text{unstretched length})$$

$$= k \times (s_C - 1 \text{ m})$$ (11)

Substituting for $\delta y_D$, $\delta s_C$, and $F_s$ from Eqs 2, 9, and 11 into the virtual-work equation, Eq. 1, gives

$$6 \cos \theta \delta \theta , \text{ by Eq. 2}$$

$$\frac{b \sin (\phi + \theta) \delta \theta}{2s_C}, \text{ by Eq. 9}$$

or

$$[(6P) \cos \theta - k(s_C - 1) \frac{b \sin (\phi + \theta)}{2s_C}] \delta \theta = 0 \quad (12)$$

Since $\delta \theta \neq 0$, the expression in brackets must equal to zero.

$$(6P) \cos \theta - k(s_C - 1) \frac{b \sin (\phi + \theta)}{2s_C} = 0$$ (12)
Eq. 12 contains the distance $s_C$, which can be calculated by using Eq. 10:

$$
(6P) \cos \theta - k \left( s_C - 1 \right) \frac{b \sin (\phi + \theta)}{2s_C} = 0
$$

(Eq. 12 repeated)

or

$$
(6P) \cos \theta - k \left( \sqrt{a - b \cos (\phi + \theta)} - 1 \right) \frac{b \sin (\phi + \theta)}{2 \sqrt{a - b \cos (\phi + \theta)}} = 0
$$

Substituting in the latter equation the values

\begin{align*}
P &= 2 \text{ kN} \hspace{1cm} \text{(Given)} \\
k &= 1.5 \text{ kN/m} \hspace{1cm} \text{(Given)} \\
a &= 15.249 \text{ m} \hspace{1cm} \text{(Eq. 7 repeated)} \\
b &= 13.416 \text{ m} \hspace{1cm} \text{(Eq. 8 repeated)} \\
\phi &= 26.565^\circ \hspace{1cm} \text{(Eq. 5 repeated)}
\end{align*}

and solving numerically gives

$$
\theta = 53.4 \hspace{1cm} \text{Ans.}
$$
11.1 Virtual Work Example 13, page 1 of 6

13. a) Determine the moment reaction at the wall F.
    b) Determine the force reaction at the roller D.
    In both cases $P = 60$ lb.

Part a. Replace the wall at F by a moment couple $M_F$ and a pin support.

Part b. Determine the force reaction at the roller D.
Consider a free-body diagram and identify the active forces associated with a small rotation of the segments of the beam.

Free-body diagram

The active "forces" are the force $P$ and the couple moment $M_F$.

Introduce coordinates $y_A$ and $\theta$ for calculating the work. 

$$\delta U = 0: \quad P \, \delta y_A - M_F \, \delta \theta = 0 \quad (1)$$
Relate the differentials $\delta y_A$ and $\delta \theta$.

By similar triangles,

$\delta y_E = \delta y_C$ and $\delta y_C = \delta y_A$

so

$\delta y_E = \delta y_A$

For small angles, $\delta y_E$ is given by

$\delta y_E = (10 \text{ ft}) \delta \theta$

But this becomes, after using the relation $\delta y_E = \delta y_A$,

$\delta y_A = 10 \delta \theta$

Substitute this result in the virtual-work equation to get

$P \delta y_A - M_F \delta \theta = 0$ (Eq. 1 repeated)

$10 \delta \theta$

Dividing through by $\delta \theta$, substituting the known value $P = 60 \text{ lb}$, and then solving for $M_F$ gives,

$M_F = 600 \text{ lb}\cdot\text{ft}$

$\leftarrow$ Ans.
11.1 Virtual Work Example 13, page 4 of 6

Part b. Replace the roller at D by a vertical force, $D_y$.

Draw a free-body diagram and show a small rotation of segments AC and CE.

Segment EF of the beam does not move because the wall support prevents both rotation and vertical displacement. Thus $M_F$ and $F_y$ do no work.

The active forces are the 60-lb force and $D_y$.

Introduce coordinates $y_A$ and $y_D$ for calculating the work. 

$$\delta U = 0: \ (60 \text{ lb}) \ \delta y_A - D_y \ \delta y_D = 0$$

(3)
11.1 Virtual Work Example 13, page 5 of 6

Relate the differentials $\delta y_A$ and $\delta y_D$.

By similar triangles,

$$\delta y_A = \delta y_C$$

and

$$\frac{\delta y_C}{5+5} = \frac{\delta y_D}{5}$$

Eliminating $\delta y_C$ gives

$$\delta y_D = \frac{\delta y_A}{2}$$

Use this equation to replace $\delta y_D$ in the virtual-work equation:

$$\left(60\right) \delta y_A - D_y \delta y_D = 0 \quad \text{(Eq. 3 repeated)}$$

Dividing through by $\delta y_A$ and solving gives

$$D_y = 120 \text{ lb} \quad \leftarrow \text{Ans.}$$
11.1 Virtual Work Example 13, page 6 of 6

15. Comment: Let's extend the discussion. If we were asked to calculate the vertical reaction force at the wall, we would replace the wall by a support that prevents rotation but permits vertical displacement.

16. Corresponding displacements

17. Segment EF translates but does not rotate. Thus the reaction moment at the support does no work. The reaction force, $F_y$, however, does work and is thus an active force.

18. Observation: The method applied in this beam example can be generalized. Virtual work can be used to calculate a force of constraint (a reaction) by considering displacements which violate the constraints and then accounting for the work done by the force of constraint. This procedure is equivalent to converting a rigid structure into a mechanism, as was done at the beginning of the present example.
11.1 Virtual Work Example 14, page 1 of 5

14. Determine the vertical reaction at support C, if \( P = 2 \) kN.

1. Convert the structure into a mechanism with one degree of freedom by replacing the pin support at C by a roller support and a vertical force, \( C_y \).
11.1 Virtual Work Example 14, page 2 of 5

2. \( C_y \) and \( P \) are active forces for the displacements shown.

3. Define coordinates \( y_B \) and \( y_C \) locating the point of application of the active forces, and compute the work.

\[
\delta U = 0: \quad -P \, \delta y_B + C_y \, \delta y_C = 0 \quad (1)
\]
11.1 Virtual Work Example 14, page 3 of 5

Relate the differentials $\delta y_B$ and $\delta y_C$ through the angles $\theta$ and $\phi$.

Begin by computing the lengths of bars BC and BA (Note that these lengths do not change, as the angles $\theta$ and $\phi$ change).

5. $BC = \sqrt{(3 \text{ m})^2 + (3 \text{ m})^2} = 3\sqrt{2} \text{ m}$

6. $AB = \sqrt{(3 \text{ m})^2 + (4 \text{ m})^2} = 5 \text{ m}$

From the above figure,

$y_B = (5 \text{ m}) \cos \theta$

$\delta y_B = -5 \sin \theta \delta \theta$ \hspace{1cm} (2)

$y_C = (5 \text{ m}) \cos \theta + (3\sqrt{2} \text{ m}) \cos \phi$

$\delta y_C = -5 \sin \theta \delta \theta - (3\sqrt{2} \text{ m}) \sin \phi \delta \phi$ \hspace{1cm} (3)

Use the law of sines to relate $\phi$ and $\theta$

$$\frac{\sin \phi}{5 \text{ m}} = \frac{\sin \theta}{3\sqrt{2} \text{ m}}$$ \hspace{1cm} (4)
11.1 Virtual Work Example 14, page 4 of 5

Differentiating Eq. 4 gives

\[ \frac{\cos \phi}{5} \delta \phi = \frac{\cos \theta}{3 \sqrt{2}} \delta \theta \]

Thus

\[ \delta \phi = \frac{5 \cos \theta \, \delta \theta}{3 \sqrt{2} \cos \phi} \quad \text{(5)} \]

The equation relating \( \delta \phi \) and \( \delta \theta \), Eq. 5, can be used in Eq. 3 to express \( \delta y_C \) in terms of \( \delta \theta \) alone:

\[ \delta y_C = -5 \sin \theta \, \delta \theta - 3 \sqrt{2} \sin \phi \, \delta \phi \]  
\[ = (-5 \sin \theta - 5 \tan \phi \cos \theta) \, \delta \theta \quad \text{(6)} \]

Substituting Eqs. 2 and 6 for \( \delta y_B \) and \( \delta y_C \) into the virtual work equation gives

\[ -5 \sin \theta \, \delta \theta, \text{ by Eq. 2} \]

\[ -P \delta y_B + C_y \delta y_C = 0 \]  
\[ = (-5 \sin \theta - 5 \tan \phi \cos \theta) \, \delta \theta, \text{ by Eq. 6} \]

\[ [P \sin \theta - C_y \sin \theta + \tan \phi \cos \theta] (5 \, \delta \theta) = 0 \]

Dividing by 5 \( \delta \theta \) gives

\[ P \sin \theta - C_y \sin \theta + \tan \phi \cos \theta = 0 \quad \text{(7)} \]
11.1 Virtual Work Example 14, page 5 of 5

Evaluating the functions of $\phi$ and $\theta$ in Eq. 7, substituting the given value $P = 2$ kN, and then solving gives

$$P \sin \theta - C_y (\sin \theta + \tan \phi \cos \theta) = 0 \quad \text{(Eq. 7 repeated)}$$

$$C_y = 0.857 \text{ kN} \quad \leftarrow \text{Ans.}$$

Observation: This example demonstrates that virtual work can be used to calculate the reaction forces from the supports acting on a structure. The example also demonstrates that just because virtual work *can* be used doesn't necessarily mean that it *should* be used—the reaction at support C could have been found much more easily by employing equilibrium equations. The usefulness of virtual work depends on how easy it is to express relations between coordinates.
11.1 Virtual Work Example 15, page 1 of 3

15. Determine the vertical reaction at support I of the truss, if $P = 10 \text{ kip} = Q$.

Convert the structure to a mechanism with one degree of freedom by replacing the pin support at I by a vertical force $I_y$ and a roller.
11.1 Virtual Work Example 15, page 2 of 3

2) Identify the active forces corresponding to a set of displacements compatible with the constraints.

3) Introduce coordinates measured from fixed points to the points of application of the applied forces.

Calculate the work done.

\[ \delta U = 0: \ P \delta y_A + Q \delta y_E - I_y \delta y_I = 0 \]  \hspace{1cm} (1)
4. Relate $\delta y_A$, $\delta y_E$, and $\delta y_I$ by geometry (similar triangles).

\[ \delta y_A = \delta y_C \]  

Similarly,

\[ \delta y_C = \delta y_E, \delta y_E = \delta y_G, \text{ and } \delta y_G = \delta y_I \]

These equations imply

\[ \delta y_A = \delta y_I \text{ and } \delta y_E = \delta y_I \]

Substituting the latter pair of equations into the virtual-work equation, Eq. 1, gives

\[ P \delta y_A + Q \delta y_E - I \delta y_I = 0 \]  

(Eq. 1 repeated)

or

\[ (P + Q - I_y) \delta y_I = 0 \]

Dividing through by $\delta y_I$, substituting the given values $P = 10 \text{ kip} = Q$, and solving gives

\[ I_y = 20 \text{ kip} \]

$\leftarrow$ Ans.
16. Determine the tension in the cord. The pulleys are frictionless and $m = 90\, \text{kg}$.

1. Convert the pulley-cord system into a mechanism with one degree of freedom by replacing the support B by a tensile force $T$ acting on the end of the cord.

Weight $= mg$
The tension $T$ and the weight do work (are active forces) if end $B$ of the cord moves up a small amount.

Introduce the coordinates $y_B$ and $y_D$.

Calculate the work done:

$$\delta U = 0: -T \delta y_B + mg \delta y_D = 0$$

(1)
11.1 Virtual Work Example 16, page 3 of 4

Relate $\delta y_B$ to $\delta y_D$ by first expressing the length, say $L$, of the cord in terms of $y_B$ and $y_D$:

$$L = [(y_D - t) - y_B] + [(y_D - t) - u - s] + [(y_D - t) - s] + \pi d_D/2 + \pi d_C/2$$

Thus

$$L = 3y_D - 3t - y_B - u - 2s + \pi d_D/2 + \pi d_C/2$$

Now differentiate, taking into account that because $L$, $t$, $u$, $s$, $d_D$, and $d_C$ do not vary as $y_D$ and $y_B$ vary, we have $\delta L = 0 = \delta t = \delta u = \delta s = \delta d_D = \delta d_C$; the result of the differentiation is, then,

$$0 = 3\delta y_D - \delta y_B$$

Thus

$$\delta y_B = 3\delta y_D \quad (2)$$
(6) Substituting this result in the virtual work equation, Eq. 1, gives

\[- T \delta y_B + mg \delta y_D = 0 \]  
\[ \text{by Eq. 2} \]

Thus

\[ (-3T + mg) \delta y_D = 0 \]

Dividing through by \( \delta y_D \), substituting \( m = 90 \text{ kg}, g = 9.81 \text{ m/s}^2 \) and solving gives

\[ T = 294 \text{ N} \]  
\[ \leftarrow \text{Ans.} \]
11.1 Virtual Work Example 17, page 1 of 9

17. Determine the equilibrium values of $\theta_1$ and $\theta_2$ for the two-bar linkage. The couple moment $M = 5 \text{ N}\cdot\text{m}$; each bar is uniform and has a mass $m$ of 5 kg; the length $L = 400 \text{ mm}$; and the unstretched length of the spring is 250 mm.

\[ k = 0.2 \text{ kN/m} \]
The system has two degrees of freedom because two coordinates \( \theta_1 \) and \( \theta_2 \) must be specified to define the position of the linkage. Consider a free-body diagram, and identify the active forces corresponding to a small change in \( \theta_1 \)—while \( \theta_2 \) is held fixed.

Because point A does not move and \( \theta_2 \) is fixed, the reactions \( A_x \) and \( A_y \), the weight \( mg \), and the spring force \( F_s \) do no work when \( \theta_1 \) is varied a small amount. Thus they are not active forces.

The couple moment \( M \) and the weight \( mg \) of the lower bar BC do work when \( \theta_1 \) is varied, so \( M \) and \( mg \) are active forces.
11.1 Virtual Work Example 17, page 3 of 9

4) In addition to the coordinate $\theta_1$, introduce a vertical coordinate $y_E$ measured downward from point A.

5) Compute the work done when the coordinates are increased a positive infinitesimal amount.

\[
dU = 0: \quad M \delta \theta_1 + mg \delta y_E = 0 \quad (1)
\]

6) Relate the differential $\delta y_E$ to the angle change, $\delta \theta_1$, by writing

\[
y_E = L \cos \theta_2 + (L/2) \cos \theta_1
\]

and then differentiating with respect to $\theta_1$, while holding $\theta_2$ fixed. That is, take the partial derivative with respect to $\theta_1$ to obtain

\[
\delta y_E = -(L/2) \sin \theta_1 \delta \theta_1 \quad (2)
\]
Substitute Eq. 2 for $y_E$ into the virtual-work equation:

$$M \delta \theta_1 + mg \delta y_E = 0$$  \hspace{1cm} (Eq. 1 repeated)

$$-(L/2) \sin \theta_1 \delta \theta_1, \text{ by Eq. 2}$$

Thus

$$[M - (mgL/2) \sin \theta_1] \delta \theta_1 = 0$$

Since $\delta \theta_1 \neq 0$, it follows that

$$M - (mgL/2) \sin \theta_1 = 0$$  \hspace{1cm} (4)

Substituting the following values into Eq. 4

$$M = 5 \text{ N-m} = 5000 \text{ N-mm}$$

$$L = 400 \text{ mm}$$

$$m = 5 \text{ kg}$$

$$g = 9.81 \text{ kg-m/s}^2$$

and solving gives

$$\theta_1 = 30.6^\circ$$  \hspace{1cm} ←Ans.
11.1 Virtual Work Example 17, page 5 of 9

8 Next identify the active forces corresponding to a small change in $\theta_2$ while $\theta_1$ is held fixed.

9 Because point A does not move, $A_x$ and $A_y$ do no work and thus are not active forces.

10 Because $\theta_1$ is fixed, link BC does not rotate. That is, the dashed line representing the new position of link BC is parallel to BC. Hence couple moment M does no work, and M is not an active force.

11 Active forces
11.1 Virtual Work Example 17, page 6 of 9

12 Introduce coordinates measured from a fixed point to the point of application of the active forces.

\[ U = 0: \ mg \delta y_E + mg \delta y_F + F_s \delta x_B = 0 \] 

13 Compute the work done.

\[ \delta U = 0: \ mg \delta y_E + mg \delta y_F + F_s \delta x_B = 0 \]
Relate the differentials $\delta y_E$, $\delta y_F$, and $\delta y_F$ to the change in angle $\delta \theta_2$:

$$x_B = L \sin \theta_2$$

$$y_F = (L/2) \cos \theta_2$$

$$y_E = L \cos \theta_2 + (L/2) \cos \theta_1$$

Differentiating each equation with respect to $\theta_2$, with $\theta_1$ held fixed, gives

$$\delta x_B = L \cos \theta_2 \delta \theta_2$$

$$\delta y_F = -(L/2) \sin \theta_2 \delta \theta_2$$

$$\delta y_E = -L \sin \theta_2 \delta \theta_2$$
11.1 Virtual Work Example 17, page 8 of 9

Substituting Eqs. 6-8 for the differentials into the virtual-work equation, Eq. 5, gives

\[-L \sin \theta_2 \delta \theta_2, \text{by Eq. 8}\]

\[mg \delta y_F + mg \delta y_E + F_s \delta x_B = 0 \quad \text{(Eq. 5 repeated)}\]

\[-(L/2) \sin \theta_2 \delta \theta_2, \text{by Eq. 7}\]

\[L \cos \theta_2 \delta \theta_2, \text{by Eq. 6}\]

Thus

\[-(3mg/2) \sin \theta_2 + F_s \cos \theta_2 \] \[L \delta \theta_2 = 0\]

Since \(L \delta \theta_2 \neq 0\), it follows that

\[-(3mg/2) \sin \theta_2 + F_s \cos \theta_2 = 0 \quad (9)\]

The force \(F_s\) in the spring can be related to \(\theta_2\):

\[F_s = k \times \text{extension of spring}\]

\[= k \times (\text{current length} - \text{unstretched length})\]

\[= k \times [(500 \text{ mm} - L \sin \theta_2) - 250 \text{ mm}]\]

\[= k(250 - L \sin \theta_2) \quad (10)\]
Substituting for $F_s$ from Eq. 10 into the virtual-work equation, Eq. 9, gives

\[-(3mg/2) \sin \theta_2 + F_s \cos \theta_2 = 0\]  
(Eq. 9 repeated)

or,

\[-(3mg/2) \sin \theta_2 + k(250 - L \sin \theta_2) \cos \theta_2 = 0\]  
(11)

Substituting the following values into Eq. 11:

$L = 400 \text{ mm}$
$m = 5 \text{ kg}$
$k = 0.2 \text{ kN/m} = 0.2 \text{ N/mm}$
$g = 9.81 \text{ kg\cdot m/s}^2$

and solving numerically gives

$\theta_2 = 18.50^\circ$  
\(\leftarrow\text{Ans.}\)